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NONLINEAR ACOUSTICS AND SHOCK WAVES  
IN SEDIMENTS AND FLUID-FILLED POROUS MEDIA

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(v) We have examined the stability of acoustic disturbances in a simple model of a porous medium with a mean flow through the pores.

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# Chapter 1

## Introduction and organization

This Final Technical Report summarizes the main results of the work performed at Rutgers under Grant N00014-94-1-0854.

### 1.1 Organization of the Report

The main technical accomplishments are outlined in Chapter 2, first in summary form, then in greater detail.

All of the work reported here has been written up in the form of papers for submission to refereed journals. The work reported in Chapter 3 below will form the basis for a paper to be submitted for publication. Chapters 4 and 5, have been accepted for publication as, respectively,

- M. A. Grinfeld and A. N. Norris, "Hamiltonian and Onsageristic approaches in the nonlinear theory of fluid permeable elastic continua", *Int. J. Engng. Sc.*
- M. A. Grinfeld and A. N. Norris, "Acoustoelasticity theory and applications for fluid-saturated porous media", *J. Acoust. Soc. Am.*

Chapter 6 has been submitted for publication, as

- M. A. Grinfeld and A. N. Norris, "Waves and flutter in fluid conveying elastic media", *J. Fluids Struct.*

One paper has appeared in print:

- A. N. Norris and M. A. Grinfeld, "Nonlinear poroelasticity for a layered medium", *J. Acoust. Soc. Am.* **98**, 1138-1146, 1995.

A copy of this is attached to the report. Two other papers have been completed on the topic of nonlinear elasticity of granular media, in collaboration with David Johnson of Schlumberger-Doll Research. They are not included here, but are cited in Section 1.2.2 below.

## 1.2 Personnel, publications and presentations

### 1.2.1 Personnel

1. PI: Andrew N. Norris

2. Dr. Michael Grinfeld

Visiting Research Professor, Department of Mechanical & Aerospace Engineering  
Rutgers University

### 1.2.2 Publications resulting from Grant N00014-94-1-0854

1. NONLINEAR POREOELASTICITY FOR A LAYERED MEDIUM, A. N. Norris and M. A. Grinfeld, *J. Acoust. Soc. Am.* **98**, 1138-1146, 1995.
2. ACOUSTOELASTICITY THEORY AND APPLICATIONS FOR FLUID-SATURATED POROUS MEDIA, M. A. Grinfeld and A. N. Norris, *J. Acoust. Soc. Am.* (accepted for publication).
3. HAMILTONIAN AND ONSAGERISTIC APPROACHES IN THE NONLINEAR THEORY OF FLUID PERMEABLE ELASTIC CONTINUA, M. A. Grinfeld and A. N. Norris, *Int. J. Engng. Sci.* (accepted for publication).
4. ACOUSTICS OF FLUID CONVEYING ELASTIC STRUCTURES, M. A. Grinfeld and A. N. Norris, *J. Fluids Struct.* (submitted).
5. NONLINEAR ELASTICITY OF GRANULAR MEDIA, A. N. Norris and D. L. Johnson, *J. Appl. Mech.* (accepted for publication). [Supported in part by N00014-94-1-0854.]
6. ROUGH ELASTIC SPHERES IN CONTACT, D. L. Johnson and A. N. Norris, *J. Mech. Phys. Solids* (submitted). [Supported in part by N00014-94-1-0854.]

### 1.2.3 Presentations at conferences

1. A.N. Norris, VARIATIONAL APPROACHES IN THE NONLINEAR THEORY OF FLUID-PERMEABLE ELASTIC CONTINUA, Acoustical Society of America, St. Louis, MO, Nov. 1995.
2. M.A. Grinfeld, ACOUSTICS OF FLUID CONVEYING ELASTIC STRUCTURES, Acoustical Society of America, Washington, D.C., June 1995
3. M.A. Grinfeld, NONLINEAR THEORY OF POREOELASTICITY AND APPLICATIONS, Third SIAM Conference on Mathematical and Computational Issues in the geosciences, San Antonio, February 1995.

## Chapter 2

# Summary of Research Accomplishments

### 2.1 Abstract Summary

Considerable progress has been made in establishing the appropriate master system of equations suitable for modeling shock waves and nonlinear wave phenomena in fluid-permeable porous solids. The major accomplishments are as follows. (i) Numerical experiments establishing the existence of shock waves in a 1D system with both Darcy and Navier-Stokes viscosities. (ii) The general form of the equations of motion have been deduced from Hamilton's principle of Least Action combined with Onsager's method of irreversible thermodynamics. (iii) We analyzed the specific case of a layered fluid/solid medium in depth, and obtained explicit formulas for the nonlinear strain energy and the permeability operator. (iv) We have extended the methods of acoustoelasticity to poroelastic media. (v) We have examined the stability of acoustic disturbances in a simple model of a porous medium with a mean flow through the pores.

### 2.2 Summary of Progress

The two major objectives in our research on shock waves in poroelastic fluid-filled media are:

1. To establish an appropriate master system (MS) of nonlinear PDEs, and
2. Analysis of physical processes within the shock transitional layer, and development of analytical and numerical tools for solving the MS.

A MS suitable for the study of nonlinear phenomena in fluid permeable elastic media (and shock wave propagation, in particular) should satisfy the following principles:

- (a) It should produce sufficiently smooth and physically acceptable finite solutions of initial-boundary-value problems for all  $t > 0$ ;
- (b) Its derivation should be conceptually simple and rely on a minimal amount of ad hoc assumptions;

- (c) It should describe all known effects of fluid-permeable elastic media;
- (d) It should include contemporary nonlinear theories one-component of fluids and solids as particular cases.

Despite numerous attempts of our predecessors in the literature we were not able to find a MS satisfying requirements (a)-(d).

We believe that we have made considerable progress in establishing the appropriate MS satisfying the "a-d" requirements. We now have a reliable framework for studying a wide variety of nonlinear phenomena of wave propagation in fluid-permeable elastic media. The principal results of our work are

- Numerical experiments establishing the existence of shock waves in a 1D system with Darcy and Navier-Stokes viscosities - see Section 2.3.1,
- The general form of the MS deduced from Hamilton's principle of Least Action combined with Onsager's method of irreversible thermodynamics - see Section 2.3.2,
- A specific form of the MS for layered media obtained by the method of homogenization - 2.3.3,
- Application and extension of the methods of acoustoelasticity to fluid-filled poro-elastic media - see Section 2.3.4,
- A study of the influence of mean flow through a porous medium on the stability of acoustical disturbances. The analysis shows that flutter-type instabilities are possible - see Section 2.3.5.

The final four items have been written up and either submitted or published [1, 2, 3, 4]. The first is still under active investigation, but will be submitted in the near future.

## 2.3 Detailed Summary of Technical Accomplishments

### 2.3.1 Shock wave structure

The key issue regarding shock waves is the question of the shock wave structure, with the following objectives: (i) To understand the interaction of physical processes within the narrow transitional layer of the shock, and (ii) to simplify the mathematical and numerical analysis by establishing the analogy of Hugoniot conditions at the shock. Mathematically, the "problem of the shock wave structure" is equivalent to the study of traveling waves with stationary profiles. Because the MS's are rather complicated we first considered the model equation (ME)  $U_t + UU_x + \Delta U + \mu U_{xx} = 0$ . The ME has many features of the MS: it is nonlinear and possesses both Darcy-like ( $\Delta U$ ) and Navier-like ( $\mu U_{xx}$ ) viscous effects.

Our analysis of traveling wave solutions for the ME indicates (surprisingly!) that the ME does not permit smooth traveling waves of finite amplitude because of the influence of the Darcy

dissipation. This finding raises several conceptual problems. Are there shock-like solutions in fluid-permeable media? If "yes" what is the analogy of the Hugoniot conditions? How should we scale the parameters of the ME and MS in order to obtain finite solutions of the shock-structure-problem? We decided to address these problems with the help of numerical experiments for the ME. Our simulations clearly show the existence of the shock-like solutions. We are currently continuing the numerical modeling of the model 1D equation in order to "experimentally" establish the Hugoniot shock conditions. For full details see Chapter 3 below.

### 2.3.2 Principle of Least Action for multi-phase media

It is common practice to formulate nonlinear equations of solids in reference or Lagrangian coordinates. Fluid dynamics, on the other hand, is more naturally considered in current or Eulerian coordinates. The reasons are clear: in solids the material particles adhere to one another, so that the reference description always provides a continuous mapping to the current configuration of particles. Not so for fluids, where large scale shearing can separate neighboring particles in the reference system. The same is true of fluid-solid composite media, because the fluid can slide past solid particles. The use of Eulerian coordinates for such media therefore appears to be the most convenient procedure and maybe the only feasible one. Despite this obvious need for an Eulerian description, the many previous works on the development of poroelastic theories are very weak on this point. They invariably obtain the equilibrium equations by postulating stresses, momenta, and various interaction effects such as a "momentum supply" for each phase.

We have taken a more fundamental point of view: that the equations governing the system should all follow unambiguously from the principle of least action or Hamilton's principle. Our starting point is a Lagrangian density per unit mass of the entire system,  $L$ , which is assumed to be a function of the following quantities for each phase  $a$  (where  $a$  can signify fluid, solid, or any one of a number of constituents in a multiphase medium): The density  $\rho_a$ , the velocity  $\mathbf{v}_a$ , and the displacement gradient  $\nabla \mathbf{u}_a$ . Conservation of mass is considered fundamental for each species, but no other balance laws are invoked in our treatment. The principle of least action is based on the stationarity of the "Action" -  $\int dt \int dV L$ , and offers the most general framework for setting up equilibrium equations and boundary conditions. There are some formidable technical difficulties involved in applying this principle in Eulerian variables and to multi-phase media. We have overcome these and have found that the general form of the equilibrium equations for a two component medium *must* be of the form

$$\rho_a \left( \frac{\partial}{\partial t} + \mathbf{v}_a \cdot \nabla \right) \frac{\rho}{\rho_a} \frac{\partial L}{\partial \mathbf{v}_a} = \operatorname{div} \mathbf{P}_a + \mathbf{G}_a, \quad a = 1 \text{ or } 2. \quad (2.1)$$

Here,  $\rho = \rho_1 + \rho_2$  is the total density, and  $\mathbf{P}_a$  is the elastic stress in  $a$  which follows from the Lagrangian density as  $P_{jia} = \delta_{ji} \rho \rho_a \frac{\partial L}{\partial \rho_a} - \rho (\delta_{ki} - \nabla_i u_{ka}) \frac{\partial L}{\nabla_j u_{ka}}$ . The vector  $\mathbf{G}_a$  is a more complicated but explicit function of the Lagrangian, but it can be interpreted as an interaction force between the two phases because of the universal identity  $\mathbf{G}_1 + \mathbf{G}_2 = 0$ .

Equation (2.1) is the main result of this part of the project. It applies to any two -phase conservative medium with a well-defined Lagrangian density. We have also generalized it to include dissipation by using a generalization of Hamilton's principle. The derivation and some applications to different models for  $L$  are in a completed paper [1] (see Chapter 4 below). Concerning the specific form of Eq. (2.1) we note that the left hand side is unanticipated and has not been previously reported. It obviously reduces to the standard mass  $\times$  acceleration for a single phase medium, either solid or fluid. The explicit formula for the interaction force is entirely new and remains to be properly understood. For full details see Chapter 4.

### 2.3.3 Homogenization: Precise equations for layered media

As a first step in understanding the MS for permeable poroelastic media we derived the equations governing finite amplitude motion in a layered fluid-solid system. The layered medium is the only configuration for which exact averaging yields explicit or semi-explicit analytical expressions for the averaged continuum. The stratified medium is also the simplest realization of a permeable porous medium, and has been previously studied both theoretically and experimentally (see [2] for references). It is rich system for linear wave motion because it displays both the Biot fast and slow waves, and they are both anisotropic.

We successfully applied the method of homogenization to derive the MS and the nonlinear constitutive equations starting from the fundamental equations of elasticity and fluid mechanics for the constituents. We find that the averaged continuum is a transversely isotropic Biot-type medium with a fully nonlinear elastic strain energy. The full details are in a paper [2] which, for the first time, presents explicit nonlinear Biot-like equations for a specific microgeometry. It also gives an exact form of the permeability convolution operator which reflects the viscous effects of the pore fluid. Previous studies have derived this in the frequency domain, which is not sufficient for application to nonlinear problems. Our time dependent permeability operator displays a new type of singularity ( $\sim 1/\sqrt{t}$ ) which must be included in shock wave models. We also examined in depth the linearized theory. Despite several previous studies, we are the first to present the complete set of linearized Biot equations for the stratified fluid/solid system. The linearized equations possess fast and slow wave solutions, which have been compared with ultrasonic experiments performed in the 1980's. This work has been published [2], and a copy is attached to the Report.

### 2.3.4 Acoustoelasticity for fluid-saturated porous media

One of the simplest nonlinear phenomena is the elastoacoustic effect whereby the speeds of small amplitude waves are changed by applying stress or strain. The theory of acoustoelasticity has been thoroughly discussed for purely elastic materials and is now being used in geoacoustics to quantify the nonlinearity of rocks and granular media (see e.g. [5]). As part of this project we derived the general theory for small dynamic motion superimposed upon large static deformation

in isotropic fluid-filled poroelastic solids. Formulae were obtained for the change in acoustic wave speeds for arbitrary loading, both on the frame and the pore fluid.

Based on the equations we obtained for the change of the fast, slow and shear wave speeds we proposed a set of specific experiments to find the complete set of third order elastic moduli for an isotropic poroelastic medium. Because of the larger number of third order moduli involved, 7 as compared with 3 for a simple elastic medium, experiments combining open-pore, closed-pore, jacketed, and unjacketed configurations are required. The details for each type of loading are presented and a set of possible experiments is discussed. Full details can be found in [3] (see Chapter 5 below).

### 2.3.5 Waves and flutter in fluid conveying elastic media

This part of the work was undertaken to understand the stability of small dynamic motions in porous media when there is a flow present in the channels. As a simple but illustrative case study, we considered a fluid flowing through a periodically layered system comprising alternating elastic solid and liquid constituents. An exact analysis was performed to consider the existence and stability of small acoustic waves and disturbances. The analysis indicates that the presence of the flow introduces the possibility of flow induced flutter. Unstable waves are generally possible for  $M$  of order unity, where  $M$  is the shear wave Mach number, but appear for much lower values for asymmetric flexural type motion. In that case it is found that a critical wavenumber exists, indicating that the system is inherently unstable to long wavelength disturbances. For full details see Chapter 6 below.

## 2.4 Current and Future Work

Now that we have established the general form of the MS in Eulerian variables [1] (Chapter 4 below) and have some sense of the form of the nonlinear elastic strain energy and permeability operators [2] we believe the work described in Section 2.3.1 is the most important. Further numerical experiments will result in better insight of the properties of the MS and shock waves in media having both Darcy-like and Navier-like dissipation. The numeric-experimental results will show us the essentially novel features of shock wave structure in fluid permeable elastic media. Our ultimate objective is to confirm these features by analytical methods as much as possible. The breakthrough with the Hugoniot conditions will open wide opportunities in understanding physical processes accompanying nonlinear wave propagation in fluid permeable elastic media - we believe that we are close to this goal.

## Chapter 3

# Shock structure in two-phase fluids

Traveling wave solutions are obtained for finite amplitude waves in a two phase liquid medium with viscous damping. The equations of motion are considered in Eulerian variables, and are based upon a recently derived Hamiltonian formulation for two phase media. Two distinct types of shocks can propagate, analogous to fast and slow waves. The speed of a 1D shock front of either type is determined by two simultaneous equations. A Hugoniot condition for the change in the internal energy is derived. A numerical example illustrates the shock profile for the two wave modes.

### 3.1 Introduction

One might suppose that the appropriate Euler-Lagrange equations must be well known for a two phase medium. But the answer is apparently in the negative. In order to understand the problem, let us first consider a 1D single phase medium, with  $x$  the current (Eulerian) coordinate,  $u(x, t)$  the particle displacement,  $L(x, u, u_x, u_t)$  the Lagrangian density per unit volume. The Euler-Lagrange equation is then

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial u_t} + \frac{\partial}{\partial x} \frac{\partial L}{\partial u_x} - \frac{\partial L}{\partial u} = 0. \quad (3.1)$$

Note that the total particle velocity is  $v = \frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x}$ , or  $v = (1 - u_x)^{-1} u_t$ . Also, the current mass density is  $\rho = \rho_0 (1 - u_x)$ , where  $\rho_0$  is the initial, undeformed density. Let  $\hat{E}(u_x)$  be the stored energy per unit mass, then the Lagrangian density per unit volume follows as

$$L = \frac{1}{2} \rho v^2 - \rho \hat{E} = \frac{\rho_0}{2} \frac{u_t^2}{1 - u_x} - \rho_0 (1 - u_x) \hat{E}. \quad (3.2)$$

Substitution of (3.2) into (3.1), yields after some simplification, the well-known equilibrium equation

$$\rho \frac{Dv}{Dt} + \frac{\partial p}{\partial x} = 0, \quad (3.3)$$

where the pressure  $p$  is defined as

$$p \equiv \rho^2 \frac{dE(\rho)}{d\rho}, \quad (3.4)$$

and  $E(\rho) = \hat{E}(u_x)$ . The point here is that simple equations are not simple to derive using Euler-Lagrange equations in current coordinates.

Another way of phrasing the issue is that the final equation, (3.4), involves quantities such as the total velocity  $v$  and the current density  $\rho$  as the natural parameters, whereas the Euler-Lagrange equations are defined by the more fundamental parameters:  $u$  and its partial derivatives with respect to space and time. In fact, the E-L equations are derived by taking independent variations of these quantities. More generally, the natural parameters of study in field theories of continuum mechanics are quantities such as the velocity  $\mathbf{v} = \mathbf{u}_t + \mathbf{v} \cdot \nabla \mathbf{u}$ , the density  $\rho = \rho(\det(\mathbf{I} - \nabla \mathbf{u}))$ , and the deformation gradient  $\nabla \mathbf{u}$  in elasticity theories<sup>1</sup>. The crucial difficulty is that  $\mathbf{v}$ ,  $\rho$  and  $\nabla \mathbf{u}$  are not independent, and it is not immediately clear how they may serve as the basis for a variational formulation. That is, the statement of the E-L equations in terms of these parameters is non-trivial, but requires tedious algebra, even though the Lagrangian  $\mathbf{L}$  is naturally defined by these variables:  $\mathbf{L} = \frac{1}{2} \rho \mathbf{v}^2 - \rho E(\rho)$ .

The situation is altogether of a different order of difficulty for two- and multi-phase media. For instance, consider a theory for a fluid-saturated porous medium in which we posit or assume two sets of physical variables at the point  $\mathbf{x}$ , see Table 1.

Table 1

Medium	Mass Density	Velocity	Strain
Solid	$\rho_s$	$\mathbf{v}_s$	$\nabla \mathbf{u}_s$
Fluid	$\rho_f$	$\mathbf{v}_f$	$\nabla \mathbf{u}_f$

Table 1. The physical parameters required for a description of a solid/fluid continuum. The subscripts  $s$  and  $f$  refer to the solid and fluid phases, respectively.

The most general form of the Lagrangian density per unit total mass is therefore of the form

$$\mathbf{L} = \mathbf{L}(\rho_s, \rho_f, \mathbf{v}_s, \mathbf{v}_f, \nabla \mathbf{u}_s, \nabla \mathbf{u}_f). \quad (3.5)$$

<sup>1</sup>Note that the deformation gradient as defined here is not the same as that normally encountered in nonlinear solid mechanics, but they may be easily related to one another.

How is one to obtain the equations of motion from this? The answer is not at all obvious, because the parameters  $\rho_s, \rho_f, \mathbf{v}_s, \mathbf{v}_f, \nabla \mathbf{u}_s$ , and  $\nabla \mathbf{u}_f$  are not independent quantities. They are related by their definitions and by the physical constraints of mass conservation for each phase, which imply the pointwise identities

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \mathbf{v}_a) = 0 \quad (3.6)$$

Recently, we have shown how to derive the governing equations by using Hamilton's principle as the starting point [1]. The derivation is detailed and unedifying, but relies upon the functional relations between  $\rho_a, \mathbf{v}_a, \nabla \mathbf{u}_a$  to derive a linear form in the virtual velocities  $\mathbf{f}_a = \frac{D\mathbf{u}_a}{D\tau} \equiv \frac{\partial \mathbf{u}_a}{\partial \tau} + \mathbf{f}_a \cdot \nabla \mathbf{u}_a$ , where  $\tau$  is the parameter of variation in the Principle of Least Action. This gives two equations: one for the solid and one for the fluid, each of the form

$$\begin{aligned} \rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} \mathbf{L}_{v_b^i} \right) &= \rho_b \nabla_i \mathbf{L} - \rho \left( \mathbf{L}_{v_b^i} \nabla_i v_b^j + \mathbf{L}_{\rho_b} \nabla_i \rho_b + \mathbf{L}_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} \right) \\ &+ \nabla_i (\rho \rho_b L_{\rho_b}) - \nabla_j (\rho L_{\nabla_j u_{kb}} A_{kib}), \quad b = s \text{ or } f, \end{aligned} \quad (3.7)$$

where

$$\frac{D_a}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_a \cdot \nabla \quad A_{.ja}^i(\mathbf{x}, t) = \delta_j^i - \nabla_j u_a^i, \quad (3.8)$$

and  $\rho = \rho_s + \rho_f$  is the total mass density of the system.

Hamilton's Principle of Least Action may be extended to account for irreversible effects, specifically, viscous body forces and viscous stresses. We also considered such irreversible effects and derived the governing equations, which turn out to be similar to (3.7) except that the right members are supplemented by the terms

$$\nabla_j P_{b.vis}^{ji} + P_{b.vis}^i, \quad b = s \text{ or } f, \quad (3.9)$$

where  $P_{s.vis}^i, P_{f.vis}^i$  are viscous forces, and  $P_{s.vis}^{ji}, P_{f.vis}^{ji}$  are viscous stresses.

In summary, these governing equations are fully general, and are suitable for nonlinear dynamic modeling of two phase media. The equations depend upon the explicit form of  $\mathbf{L}$  used, but remain too general and too complex for studying shock structure in fluid/solid continua. Instead, we consider here the simpler two phase medium: a one dimensional model of a two liquid medium.

The two liquid medium still displays many of the most relevant characteristics of fluid-saturated porous solids, such as the existence of two distinct compressional wave types for infinitesimal motions. The structure of shocks waves in uniform fluids is well understood by now, e.g. [49], but there is no corresponding theory available for systems of two or more phases (models exist which include the influence of subsidiary phases, such as secondary gases in gas mixtures, but they are, as far as we understand them, essentially single phase theories). Among our objectives here are the following: (i) to determine if two types of shocks are possible (fast and slow), (ii) to understand the dependence of shock speed on amplitude, and (iii) to understand the role of viscous effects. The permeability is another important parameter in the dynamics of porous media, and can be included here as a dissipative (viscous) term. However, only 'direct' viscosity, will be considered here. Also, we ignore entropy effects in this paper.

### 3.2 Governing 1D equations of a liquid mixture

The physical quantities of interest are the densities  $\rho_1, \rho_2$ , and the particle velocities  $v_1, v_2$ , all of which are functions of time and the single spatial coordinate  $x$ . The total mass density at a point is  $\rho = \rho_1 + \rho_2$ . The equations of mass conservation for each species are

$$\frac{D_1 \rho_1}{Dt} + \rho_1 \frac{\partial v_1}{\partial x} = 0, \quad \frac{D_2 \rho_2}{Dt} + \rho_2 \frac{\partial v_2}{\partial x} = 0, \quad (3.10)$$

where the total material derivatives for each phase are

$$\frac{D_1}{Dt} = \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x}, \quad \frac{D_2}{Dt} = \frac{\partial}{\partial t} + v_2 \frac{\partial}{\partial x}. \quad (3.11)$$

The governing equations of motion may be expressed succinctly in terms of Lagrangian density per unit total mass  $L(v_1, v_2, \rho_1, \rho_2)$ , or explicitly

$$L = T(v_1, v_2, \rho_1, \rho_2) - E(\rho_1, \rho_2), \quad (3.12)$$

where  $E$  is the elastic energy function, and  $T$  the kinetic energy. Thus, from eqs. (3.7)-(3.9) and [1],

$$\rho_1 \frac{D_1}{Dt} \left( \frac{\rho}{\rho_1} T_{v_1} \right) = -\frac{\partial p_1}{\partial x} + F_{12} + \mu_1 \frac{\partial^2 v_1}{\partial x^2}, \quad (3.13)$$

$$\rho_2 \frac{D_2}{Dt} \left( \frac{\rho}{\rho_2} T_{v_2} \right) = -\frac{\partial p_2}{\partial x} + F_{21} + \mu_2 \frac{\partial^2 v_2}{\partial x^2}, \quad (3.14)$$

where the partial pressures are

$$p_1 = -\rho \rho_1 L_{\rho_1}, \quad p_2 = -\rho \rho_2 L_{\rho_2}, \quad (3.15)$$

and  $F_{12}, F_{21}$  may be interpreted as equal but opposite interaction force densities,

$$F_{12} = -F_{21} = \rho_1 L_{v_2} \frac{\partial v_2}{\partial x} + \rho_1 L_{\rho_2} \frac{\partial \rho_2}{\partial x} - \rho_2 L_{v_1} \frac{\partial v_1}{\partial x} - \rho_2 L_{\rho_1} \frac{\partial \rho_1}{\partial x}. \quad (3.16)$$

The total mechanical energy density (elastic plus kinetic) is [1]

$$\mathcal{E} = v_1 L_{v_1} + v_2 L_{v_2} - L, \quad (3.17)$$

and the total energy flux at a point is [1]

$$\mathcal{F} = p_1 v_1 + p_2 v_2 + (v_1 - v_2) \rho_2 v_1 L_{v_1} + (v_2 - v_1) \rho_1 v_2 L_{v_2}. \quad (3.18)$$

The equation for the balance of energy at a point follows from the equations of motion as [1]

$$\rho_1 \frac{D_1 \mathcal{E}}{Dt} + \rho_2 \frac{D_2 \mathcal{E}}{Dt} + \frac{\partial \mathcal{F}}{\partial x} = -\mu_1 \left( \frac{\partial v_1}{\partial x} \right)^2 - \mu_2 \left( \frac{\partial v_2}{\partial x} \right)^2. \quad (3.19)$$

### 3.3 Propagating waves

#### 3.3.1 Shock speed and modes

Consider a propagating wave of the form  $f(x, t) = f(\xi)$  where  $\xi = x - ct$  for any variable  $f$ . Thus,  $D_a f / D t = (v_a - c)f'$ , and so the mass conservation equations become

$$(c - v_1)\rho_1 = a_1, \quad (c - v_2)\rho_2 = a_2, \quad (3.20)$$

where  $a_1$  and  $a_2$  are constants. Using (3.20) to simplify  $F_{12}$  and  $F_{21}$ , the equations of motion (3.13) become

$$-P'_1 + P_1 \frac{\rho_2 \rho'_1}{\rho \rho_1} - P_2 \frac{\rho_1 \rho'_2}{\rho \rho_2} + \mu_1 v''_1 = 0, \quad (3.21)$$

$$-P'_2 - P_1 \frac{\rho_2 \rho'_1}{\rho \rho_1} + P_2 \frac{\rho_1 \rho'_2}{\rho \rho_2} + \mu_2 v''_2 = 0. \quad (3.22)$$

where

$$P_a \equiv p_a - a_a \frac{\rho}{\rho_a} T_{v_a}. \quad (3.23)$$

Note that

$$P_a = -\rho \rho_a \hat{L}_{\rho_a}, \quad (3.24)$$

where

$$\hat{L}(\rho_1, \rho_2) \equiv L \left( c - \frac{a_1}{\rho_1}, c - \frac{a_2}{\rho_2}, \rho_1, \rho_2 \right), \quad (3.25)$$

is a function only of  $\rho_1$  and  $\rho_2$  because  $v_1$  and  $v_2$  have been eliminated using (3.20).

Adding the two equations of motion and integrating, implies that

$$\mu_1 v'_1 + \mu_2 v'_2 - P_1 - P_2 = b, \quad (3.26)$$

where  $b$  is another constant. The constants can be determined from the conditions at  $\xi = \pm\infty$ . The wave is assumed to propagate into an undisturbed medium, implying  $v_a \rightarrow 0$ , as  $\xi \rightarrow \infty$ , and hence

$$a_1 = \rho_{1\infty} c, \quad a_2 = \rho_{2\infty} c, \quad b = -p_{1\infty} - p_{2\infty}, \quad (3.27)$$

where  $\rho_{1\infty}$ ,  $\rho_{2\infty}$  and  $p_{1\infty}$ ,  $p_{2\infty}$ , are the equilibrium densities and partial pressures ahead of the wave. Normally, and in the following, both  $p_{1\infty}$  and  $p_{2\infty}$  are zero. The densities at any point can now be expressed in terms of the velocities using equations (3.20) and (3.27),

$$\rho_1 = \frac{\rho_{1\infty} c}{c - v_1}, \quad \rho_2 = \frac{\rho_{2\infty} c}{c - v_2}, \quad (3.28)$$

or

$$v_1 = (\rho_1 - \rho_{1\infty}) \frac{c}{\rho_1}, \quad v_2 = (\rho_2 - \rho_{2\infty}) \frac{c}{\rho_2}. \quad (3.29)$$

Also, note that

$$P_a = p_a - \rho_\infty c T_{v_a}, \quad (3.30)$$

where  $\rho_\infty = \rho_{1\infty} + \rho_{2\infty}$ . Integrating the fundamental equations (3.21) yields

$$P_1(v_{1,sh}, v_{2,sh}) = p_{1\infty} + I_W, \quad P_2(v_{1,sh}, v_{2,sh}) = p_{2\infty} - I_W, \quad (3.31)$$

where

$$\begin{aligned} I_W &= - \int_{-\infty}^{\infty} F_{12} d\xi = \int_{-\infty}^{\infty} \left( P_2 \frac{\rho_1 \rho'_2}{\rho \rho_2} - P_1 \frac{\rho_2 \rho'_1}{\rho \rho_1} \right) d\xi \\ &= \int_{-\infty}^{\infty} \left( \rho_2 \hat{L}_{\rho_1} \rho'_1 - \rho_1 \hat{L}_{\rho_2} \rho'_2 \right) d\xi. \end{aligned} \quad (3.32)$$

The pair of equations (3.31) provides a nonlinear eigenvalue system for the wave speed  $c$  and the particle velocities  $v_{1,sh}$  and  $v_{2,sh}$ . The system (3.31) as it stands is not closed but requires prior knowledge of the integral of the interaction force  $I_W$ .

### 3.3.2 A Hugoniot relation

When applied to traveling waves the energy balance (3.19) becomes, using the identities (3.29),

$$-\rho_\infty c \mathcal{E}' + \mathcal{F}' = -\mu_1 (v'_1)^2 - \mu_2 (v'_2)^2. \quad (3.33)$$

Integrating through the shock yields

$$\rho_\infty c \mathcal{E}_{sh} - \rho_\infty c \mathcal{E}_\infty = \mathcal{F}_{sh} + D_{vis}, \quad (3.34)$$

where  $D_{vis}$  is the total viscous dissipation for the shock,

$$D_{vis} \equiv \int_{-\infty}^{\infty} \left[ \mu_1 (v'_1)^2 + \mu_2 (v'_2)^2 \right] d\xi. \quad (3.35)$$

The elastic energy density ahead of the shock is assumed to be zero with no loss in generality,  $\mathcal{E}_\infty = 0$ . Also, the flux far behind the shock,  $\mathcal{F}_{sh}$ , can be simplified using the characteristic equations (3.31). Thus, (3.34) becomes

$$\begin{aligned} \rho_\infty c \mathcal{E}_{sh} - D_{vis} &= \rho_{1\infty} c v_{1,sh} T_{v_1} + \rho_{2\infty} c v_{2,sh} T_{v_2} \\ &\quad + (v_1 - v_2) (\rho_{2,sh} v_{1,sh} T_{v_1} - \rho_{1,sh} v_{2,sh} T_{v_2} + I_W). \end{aligned} \quad (3.36)$$

The left member reflects the total change in internal energy density caused by the passage of the shock. This Hugoniot-type relation also connects the integral  $I_W$  with the viscous dissipation integral  $D_{vis}$ .

### 3.4 Shock structure

The structure of the shock follows from the ODEs (3.21). Alternatively, taking a linear superposition of the two equations of motion (3.21) gives

$$\mu_1\mu_2(v''_1 - v''_2) + (\mu_1 + \mu_2) \left( P_1 \frac{\rho_2 \rho'_1}{\rho \rho_1} - P_2 \frac{\rho_1 \rho'_2}{\rho \rho_2} \right) + \mu_1 P'_2 - \mu_2 P'_1 = 0. \quad (3.37)$$

We will use the alternative pair of ODEs (3.26) and (3.37) which define a third order system as compared with the original fourth order equations (3.21). Define

$$U = \mu_1 v_1 + \mu_2 v_2, \quad (3.38)$$

$$V = v_1 - v_2 \quad (3.39)$$

$$W = v'_1 - v'_2 - \frac{P_1}{\mu_1} + \frac{P_2}{\mu_2}, \quad (3.40)$$

then we may rewrite equation (3.37) as

$$W'(\xi) = (\mu_1^{-1} + \mu_2^{-1}) \left( P_2 \frac{\rho_1 \rho'_2}{\rho \rho_2} - P_1 \frac{\rho_2 \rho'_1}{\rho \rho_1} \right). \quad (3.41)$$

The derivatives  $\rho'_1$  and  $\rho'_2$  can be expressed in terms of derivatives of  $U$  and  $V$  by using (3.28) and the relations  $v_1 = (U + \mu_2 V)/(\mu_1 + \mu_2)$ ,  $v_2 = (U - \mu_1 V)/(\mu_1 + \mu_2)$ . The equations therefore reduce to a system of three first order ODEs:

$$U'(\xi) = P_1 + P_2 - p_{1\infty} - p_{2\infty}, \quad (3.42)$$

$$V'(\xi) = \frac{P_1}{\mu_1} - \frac{P_2}{\mu_2} + W, \quad (3.43)$$

$$W'(\xi) = \frac{\rho_1 \rho_2}{\mu_1 \mu_2 \rho c} \left[ \frac{P_2}{\rho_{2\infty}} (U' - \mu_1 V') - \frac{P_1}{\rho_{1\infty}} (U' + \mu_2 V') \right]. \quad (3.44)$$

Behind the wave the velocities attain non-zero asymptotic values, Thus,  $v_a \rightarrow v_{a,sh}$  as  $\xi \rightarrow -\infty$ . The densities far behind the wave follow from (3.28) as

$$\rho_{1sh} = \frac{\rho_{1\infty} c}{c - v_{1,sh}}, \quad \rho_{2sh} = \frac{\rho_{2\infty} c}{c - v_{2,sh}}. \quad (3.45)$$

Equations (3.42) and (3.43) therefore imply the identities

$$P_1(v_{1,sh}, v_{2,sh}) + P_2(v_{1,sh}, v_{2,sh}) = p_{1\infty} + p_{2\infty}, \quad (3.46)$$

$$\frac{P_1(v_{1,sh}, v_{2,sh})}{\mu_1} - \frac{P_2(v_{1,sh}, v_{2,sh})}{\mu_2} = \frac{p_{1\infty}}{\mu_1} - \frac{p_{2\infty}}{\mu_2} + \lim_{\xi \rightarrow \infty} W - \lim_{\xi \rightarrow -\infty} W. \quad (3.47)$$

This pair of equations can be rewritten as (3.31) because of the identity

$$I_W = (\mu_1^{-1} + \mu_2^{-1})^{-1} \int_{-\infty}^{\infty} W'(\xi) d\xi. \quad (3.48)$$

In the linearized limit, the terms  $p_{1\infty}$ ,  $p_{2\infty}$  and  $W$  are second order, and therefore the eigenvalue system reduces to

$$P_1(v_{1,sh}, v_{2,sh}) = 0, \quad P_2(v_{1,sh}, v_{2,sh}) = 0. \quad (3.49)$$

Explicitly linearizing yields a pair of simultaneous equations

$$\begin{bmatrix} \rho_1^2 E_{\rho_1 \rho_1} - c^2 T_{v_1 v_1} & \rho_1 \rho_2 E_{\rho_1 \rho_2} - c^2 T_{v_1 v_2} \\ \rho_1 \rho_2 E_{\rho_2 \rho_1} - c^2 T_{v_2 v_1} & \rho_2^2 E_{\rho_2 \rho_2} - c^2 T_{v_2 v_2} \end{bmatrix} \begin{bmatrix} \rho v_{1,sh} \\ \rho v_{2,sh} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (3.50)$$

### 3.4.1 Single phase medium

For a single phase medium, equation (3.42) becomes "Burger's" equations, that is,

$$\mu v' = -p_\infty - \rho^2 \hat{L}_\rho, \quad (3.51)$$

or in terms of  $\rho$  alone,

$$\mu \rho_\infty c \frac{\rho'}{\rho^2} = p(\rho) - p_\infty - \rho_\infty c^2 \frac{(\rho - \rho_\infty)}{\rho}. \quad (3.52)$$

Equation (3.52) can be integrated as

$$\xi = \mu \int_{\eta(0)}^{\eta} \left[ \frac{\eta_\infty}{c} (p_\infty - p) + \frac{c}{\eta_\infty} (\eta_\infty - \eta) \right]^{-1} d\eta, \quad (3.53)$$

where  $\eta = 1/\rho$ .

The characteristic equation is the single condition (3.46), which becomes (see p. 159 of [50])

$$c^2 = \frac{\rho_{sh}}{\rho_\infty} \left( \frac{p_{sh} - p_\infty}{\rho_{sh} - \rho_\infty} \right), \quad v_{sh}^2 = \frac{(p_{sh} - p_\infty)(\rho_{sh} - \rho_\infty)}{\rho_{sh} \rho_\infty}. \quad (3.54)$$

Therefore, eliminating  $c$  from (3.53) yields

$$\frac{\xi}{\mu} = \int_{\rho(0)}^{\rho} \frac{\sqrt{\rho_{sh} \rho_\infty (p_{sh} - p_\infty)(\rho_{sh} - \rho_\infty)} d\rho}{\rho^2 (p(\rho) - p_\infty)(\rho_{sh} - \rho_\infty) - \rho \rho_{sh} (p_{sh} - p_\infty)(\rho - \rho_\infty)}. \quad (3.55)$$

The constant of integration can be eliminated by defining  $\xi = 0$  to be the point at which  $\rho$  has a certain value between  $\rho_\infty$  and  $\rho_{sh}$ . For instance, we may set  $\rho(0) = \frac{1}{2}(\rho_{sh} + \rho_\infty)$ , then (3.55) yields the shock profile.

As a first approximation to the exact shock structure, the denominator in the integral can be replaced by a polynomial with zeros as  $\rho = \rho_{sh}$  and  $\rho = \rho_\infty$ . The simplest approximation is a quadratic,

$$\rho^2(p(\rho) - p_\infty)(\rho_{sh} - \rho_\infty) - \rho\rho_{sh}(p_{sh} - p_\infty)(\rho - \rho_\infty) \approx \rho_{sh}(p_{sh} - p_\infty)(\rho - \rho_{sh})(\rho - \rho_\infty), \quad (3.56)$$

which can be integrated exactly, to give

$$\rho(\xi) = \frac{1}{2}(\rho_{sh} + \rho_\infty) - \frac{1}{2}(\rho_{sh} - \rho_\infty) \tanh\left((\rho_{sh} - \rho_\infty)\frac{c\xi}{2\mu}\right). \quad (3.57)$$

### 3.4.2 Linearized equations

In the linear approximation the terms involving  $F_{12}$  and  $F_{21}$  of (3.16) are of second order in the small quantities: the deviations in the densities and the particle velocities. Let  $v_1$  and  $v_2$  be the independent variables, then the linearized equations are, from (3.21),

$$\mu_1 v_1'' + \left(cT_{v_1 v_1} - \frac{\rho_1^2}{c} E_{\rho_1 \rho_1}\right) \rho v_1' + \left(cT_{v_1 v_2} - \frac{\rho_1 \rho_2}{c} E_{\rho_1 \rho_2}\right) \rho v_2' = 0, \quad (3.58)$$

$$\mu_2 v_2'' + \left(cT_{v_1 v_2} - \frac{\rho_1 \rho_2}{c} E_{\rho_1 \rho_2}\right) \rho v_1' + \left(cT_{v_2 v_2} - \frac{\rho_2^2}{c} E_{\rho_2 \rho_2}\right) \rho v_2' = 0. \quad (3.59)$$

## 3.5 Numerical experiments

The numerical problem can be cast as three coupled first order ODE's for  $v_1$  and  $v_2$ , defined by eq. (3.26) with  $b = 0$  and eq. (3.37). This presents a nonlinear eigenvalue problem for  $c$ , and the values of  $v_1$  and  $v_2$  (or  $\rho_1$  and  $\rho_2$ ) behind the shock are  $v_1 \rightarrow v_{1,sh}$ ,  $v_2 \rightarrow v_{2,sh}$ , as  $\xi \rightarrow -\infty$ . The nonlinear eigenvalue problem can be expressed in almost algebraic form via eqs. (3.31) with  $p_{1\infty} = 0$  and  $p_{2\infty} = 0$ . The difficulty is due to the term  $I_W$ , which is the total integrated interaction force through the shock. For single phase:  $I_W = 0$  and we immediately have an algebraic equation for  $c = c(v_{sh})$ . For two phases:  $c = c(v_{1,sh}, v_{2,sh}, I_W)$ , which requires explicit numerical solution. For infinitesimal motion, we have  $I_W = 0$ , and thus obtain two linear equations for  $v_{1,sh}$ ,  $v_{2,sh}$  (Biot-type equations) with fast and slow roots for  $c$ .

Numerical experiments were performed to solve for the shock speed and structure. The procedure used (so far) is to solve the ODES iteratively. First, fix  $v_1(0)$ , guess  $v_2(0)$  and  $c$ . Then integrate from  $\xi = 0$  both ways, and subsequently adjust  $v_2(0)$  and  $c$  to iteratively reach steady states at large  $\xi$ . We find that there can exist two types of shock fronts: fast and slow, with the following characteristics. the fast shock has  $v_{1,sh} v_{2,sh} > 0$  (in phase), while the slow shock has  $v_{1,sh} v_{2,sh} < 0$  (out of phase). A typical pair of solutions is shown in Figure 3.1.

## Chapter 4

# Hamiltonian methods for fluid-saturated porous media

A new system of equations governing nonlinear dynamics of fluid permeable poro-elastic media is derived on the basis of Hamilton's principle for reversible effects and the Onsager-Sedov approach for irreversible effects. The equations are in Eulerian variables, suitable for dealing with fluid and solid phenomena simultaneously. The classical (Murnaghan-like) equations of nonlinear elasticity, as well as the governing equations of the ideal and Navier-Stokes fluids are shown to be special cases of the general governing system. It is well-known that the Navier-Stokes equations of compressible fluid provide a correct self-consistent basis for studying a wide variety of nonlinear effects in fluids. The authors believe that the governing system proposed here provides the same opportunities for various nonlinear effects in poro-elastic fluid-penetrable media.

### 4.1 Introduction

It is common practice to formulate nonlinear equations of solids in reference or Lagrangian coordinates. Fluid dynamics, on the other hand, is more naturally considered in current or Eulerian coordinates. The reasons are clear: in solids the material particles adhere to one another, so that the reference description always provides a continuous mapping to the current configuration of particles. Not so for fluids, where large scale shearing can separate neighboring particles in the reference system. The same is true of fluid-solid composite media, because the fluid can slide past solid particles. The use of Eulerian coordinates for such media therefore appears to be the most convenient procedure, and maybe the only feasible one.

The purpose of this Chapter is to establish the governing equations for fluid-solid media in Eulerian coordinates by means of Hamilton's principle for conservative media and the Onsager-Sedov technique for irreversible effects. The idea is to start with a well defined Lagrangian density, from which all equations should follow. Although this approach is potentially the most reliable and flexible its usage for the Eulerian description of continuous media demands some care. However, we believe this is the most general and consistent approach to formulating nonlinear equations of fluid-solid continua. In particular, it allows one to study the internal structure of

shock waves and flutter-like phenomena, among others.

We note that there are many excellent articles on the fundamental equations of fluid-permeable solids. Nonlinear treatments are usually framed in terms of a mixture theory [41, 42] for which the starting point is a set of balance laws for mass, momentum, *etc.* Here we take one step back and invoke a stationarity principle, that of least action, to generate the equilibrium equations. Conservation of mass is considered fundamental for each species, but no other balance laws are invoked. Mixture theories can also be formulated from an energy stand point [43], and nonlinear poroelasticity has been developed from a thermodynamic point of view [44]. However, we are unaware of a treatment based on the Hamiltonian principle of least action as developed here.

In many cases discrete and distributed mechanical systems show a behavior which can be treated as conservative. It seems reasonable to study such behavior using a proper generalization of the Hamilton variational principle. Following some preliminary definitions in Section 2 we begin in Section 3 by considering the procedure of [45]-[47] for performing variations in Eulerian variables. The remainder of Section 3 describes the derivation of the governing equations for a conservative system from Hamilton's principle. We build upon these results in Section 4 and include dissipative or irreversible effects, associated with fluid viscosity for instance. The starting point is a generalization of Hamilton's principle using the Onsager-Sedov method. Some general results concerning energy are discussed. Finally in Section 5 we provide a variational derivation of the dynamic equations of classical hydrodynamics and of Murnaghan's equations of nonlinear elasticity [48]. We also present three examples of the application of the governing equations to two component media included fluid-filled poroelastic solids.

## 4.2 Notation and the Eulerian description of two-component continua

The use of the Eulerian description has several advantages for analyzing two-component media, and fluid permeable solids in particular. In the Eulerian description time  $t$  and the coordinates  $z^i$  (Latin indices like  $i, j, k, l$  take the values 1, 2, 3) are used as independent variables. For the sake of simplicity the reader can think that the system is referred to the Cartesian coordinates although we prefer to use a covariant form of the equations (in particular, this form makes general equations much more eloquent and expressive, and it allows one to switch easily between geometrically different coordinate systems). We use the notation  $z_{ij}, z^{ij}, \nabla_i$  for the co- and contra-variant metrics and covariant differentiation (in the Cartesian system  $z_{ij} = z^{ij} = \delta_j^i$  and co- and contra-variant components appear to be equal while covariant differentiation reduces to the partial  $\nabla_i = \partial/\partial z^i$ ). In order to distinguish between different characteristics of the two components we use extra-subscripts  $a, b$  taking values 1 or 2 or the subscripts  $s$  or  $f$  when emphasizing the solid or fluid nature of the component. In particular, we use the notation  $u_a^i(z, t)$ ,  $\rho_a(z, t)$ , and  $v_a^i(z, t)$  for the displacements, densities and velocities of finite deformations of the components. Introducing the "material" time derivative  $D_a/Dt = \partial/\partial t + v_a^i \nabla_i$  associated with the "a-th" constituent we get by definition

$$v_a^i(z, t) = \frac{D_a u_a^i}{Dt} = \left( \frac{\partial}{\partial t} + v_a^j \nabla_j \right) u_a^i. \quad (4.1)$$

The following pair of identities are immediate consequences of Eq. (4.1)

$$\frac{\partial u_a^i(z, t)}{\partial t} = v_a^j A_{.ja}^i, \quad v_a^i(z, t) = \frac{\partial u_a^j(z, t)}{\partial t} B_{.ja}^i, \quad (4.2)$$

where  $A_{.ja}^i(z, t) = \delta_j^i - \nabla_j u_a^i$  and  $B_{.ja}^i(z, t)$  is the inverse of  $A_{.ja}^i$ , i.e.,  $B_{.ja}^i A_{.ka}^j = \delta_k^i$ .

The fields  $\rho_a(z, t)$  and  $v_a^i(z, t)$  obey the well-known formula of mass conservation for each phase,

$$\frac{\partial \rho_a}{\partial t} + \nabla_j(\rho_a v_a^j) = 0. \quad (4.3)$$

The differential form of the mass conservation result (4.3) is equivalent to the following algebraic one

$$\rho_a = \rho_a^0 \det|A_{.ja}|, \quad (4.4)$$

where  $\rho_a^0$  is the mass density in the reference configuration. Differentiating Eq. (4.4) we get the following relationship

$$\frac{\partial \rho_a}{\partial \nabla_i u_a^i} = -\rho_a B_{.ja}^i. \quad (4.5)$$

### 4.3 The Hamiltonian approach for two-component media in the Eulerian variables

Assuming the absence of external mass and surface forces we choose the Action  $S$  corresponding to a “cold” multi-component substance in the following standard form:

$$S = \int_{t_0}^{t_1} dt \int_{\Omega} d\Omega \rho(z, t) L, \quad (4.6)$$

where  $L = T - E$  and  $T$  and  $E$  are the kinetic and internal (elastic) energies of the substance per unit mass, and  $\rho(z, t)$  is the mass density of the substance which is equal to the sum of the partial densities  $\rho_a(z, t)$  of the constituents

$$\rho(z, t) = \sum_a \rho_a(z, t). \quad (4.7)$$

In what follows we assume that the kinetic energy density is an algebraic function of velocities of the constituents  $v_a^i(z, t)$  while the internal energy density is a function of the displacement gradients  $\nabla_j u_a^i(z, t)$  and the partial densities  $\rho_a(z, t)$ :

$$T = T(v_a^i), \quad E = E(\nabla_j u_a^i, \rho_a), \quad L = L(\rho_a, v_a^i, \nabla_j u_a^i). \quad (4.8)$$

The present analysis applies to any number of constituents,  $a = 1, 2, \dots$ , but for practical purposes we will be concerned with two-component media.

Dynamic equations and natural boundary conditions of the actual motion can be obtained as the conditions of vanishing of the first variation of the Action  $S$  for all “admissible” trajectories

$(\rho_a(z, t), v_a^i(z, t), u_a^i(z, t))$ . In addition to some natural demands of sufficient smoothness, all admissible fields have to obey certain ‘mechanical’ constraints. First of all, as always, all admissible dynamical fields have to coincide with the actual fields in the initial (at  $t = t_0$ ) and final (at  $t = t_1$ ) configurations. Other crucial constraints follow from the fact that the fields  $(\rho_a, v_a^i, u_a^i)$  are mutually dependent since, say, the two former fields can be found from the latter by means of spatial and time differentiation. This fact implies certain linear differential relationships for the variations of these fields, and the independent variations should be carefully extracted in the first variation of the Action  $\delta S$ . In order to make this point more transparent let us consider a one-parameter family of the admissible fields

$$\rho_a = \rho_a(z, t, \tau), \quad v_a^i = v_a^i(z, t, \tau), \quad u_a^i = u_a^i(z, t, \tau), \quad (4.9)$$

where  $\tau$  is the parameter of variation. By definition, the Eulerian variations of these fields are the following functions of  $z$  and  $t$ :

$$\begin{aligned} \delta \rho_a(z, t) &= \left. \frac{\partial \rho_a}{\partial \tau}(z, t, \tau) \right|_{\tau=0} = R_a(z, t), \\ \delta v_a^i(z, t) &= \left. \frac{\partial v_a^i}{\partial \tau}(z, t, \tau) \right|_{\tau=0} = G_a^i(z, t), \\ \delta u_a^i(z, t) &= \left. \frac{\partial u_a^i}{\partial \tau}(z, t, \tau) \right|_{\tau=0} = Q_a^i(z, t). \end{aligned} \quad (4.10)$$

With the help of Eqs. (4.2), (4.4) and (4.5) one can easily establish the following explicit formulae for the variations  $R_a(z, t)$ ,  $G_a^k(z, t)$  in terms of the variation  $Q_a^i(z, t)$  and its derivatives:

$$G_a^k(z, t) = B_{.ma}^k \left( \frac{\partial Q_a^m}{\partial t} + \frac{\partial u_a^j}{\partial t} B_{.ja}^n \nabla_n Q_a^m \right), \quad R_a(z, t) = -\rho_a B_{.ia}^j \nabla_j Q_a^i. \quad (4.11)$$

In order to establish a system of governing equations one ought to substitute the one-dimensional fields (4.9) into (4.6), differentiate the integral, and then, extract the independent variation  $Q_a^i$  using the relationships (4.11) and integration by parts (in space and time). Then, equating to zero the coefficients of  $Q_a^i$  in the integrand we arrive at the desired master system. However, this approach, although absolutely correct conceptually, requires very laborious computation when the Eulerian description is used. In what follows we use a somewhat different technique which seems to be much more convenient when dealing with the Eulerian description. In this technique the virtual velocities  $f_a^i(z, t, \tau)$  of the constituents are treated as independent variations rather than the functions  $\delta u_a^i(z, t)$ . The virtual velocities  $f_a^i(z, t, \tau)$  are defined implicitly by the relationship

$$f_a^i(z, t, \tau) = \frac{D_a}{D\tau} u_a^i(z, t, \tau), \quad (4.12)$$

where  $D_a/D\tau$  is the material derivative associated with the parameter  $\tau$ :  $D_a/D\tau = \partial/\partial\tau + f_a^i \nabla_i$ .

The following statement is crucial for overcoming future computational obstacles.

LEMMA. The operators  $D_a/Dt$  and  $D_a/D\tau$  commute on  $u_a^k(z, t, \tau)$

$$\left[ \frac{D_a}{Dt}, \frac{D_a}{D\tau} \right] u_a^i = \left( \frac{D_a}{Dt} \frac{D_a}{D\tau} - \frac{D_a}{D\tau} \frac{D_a}{Dt} \right) u_a^i = \frac{D_a f_a^i}{Dt} - \frac{D_a v_a^i}{D\tau} = 0. \quad (4.13)$$

PROOF. Differentiating Eq. (4.2) we get

$$\frac{D_a v_a^i}{D\tau} = \frac{D_a}{D\tau} \left( \frac{\partial u_a^l}{\partial t} B_{.la}^i \right) = \frac{D_a}{D\tau} \left( \frac{\partial u_a^l}{\partial t} \right) B_{.la}^i + \frac{\partial u_a^l}{\partial t} \frac{D_a B_{.la}^i}{D\tau}. \quad (4.14)$$

One can easily derive the following relationships:

$$\frac{D_a B_{.la}^i}{D\tau} = -B_{.na}^i B_{.la}^m \frac{D_a A_{.ma}^n}{D\tau},$$

$$\frac{D_a A_{.ma}^n}{D\tau} = -\frac{D_a \nabla_m u_a^n}{D\tau} = -A_{.ja}^n \nabla_m f_a^j,$$

$$\frac{D_a}{D\tau} \frac{\partial u_a^l}{\partial t} = \frac{\partial}{\partial t} \frac{D_a u_a^l}{D\tau} - \frac{\partial f_a^j}{\partial t} \nabla_j u_a^l = \frac{\partial f_a^j}{\partial t} A_{.ja}^l.$$

Inserting these in the RHS of (4.14) we arrive at the desired result

$$\frac{D_a v_a^i}{D\tau} = \frac{\partial f_a^i}{\partial t} + (\nabla_m f_a^i) B_{.la}^m \frac{\partial u_a^l}{\partial t} = \frac{\partial f_a^i}{\partial t} + v_a^m \nabla_m f_a^i = \frac{D_a f_a^i}{Dt}$$

We remind the reader of the following formulae for a derivative of the integral over a domain occupied by a moving body consisting of particles of the same material:

$$\frac{d}{d\tau} \int_{\Omega} d\Omega \rho_a L = \int_{\Omega} d\Omega \rho_a \frac{D_a L}{D\tau}, \quad \frac{d}{dt} \int_{\Omega} d\Omega \rho_a L = \int_{\Omega} d\Omega \rho_a \frac{D_a L}{Dt}. \quad (4.15)$$

Combining (4.6) and (4.15) we get the following formula

$$\frac{dS}{d\tau} = \int_{t_0}^{t_1} dt \frac{\partial}{\partial \tau} \int_{\Omega} d\Omega \sum_a \rho_a L = \int_{t_0}^{t_1} dt \int_{\Omega} d\Omega \sum_a \rho_a \frac{D_a L}{D\tau}. \quad (4.16)$$

The fundamental equations of the system follow by equating the latter to zero, which we will now proceed to do. First, we note the identity

$$\frac{D_a}{D\tau} = \frac{D_b}{D\tau} - (f_b^k - f_a^k) \nabla_k, \quad (4.17)$$

which combined with (4.13) yields the relationships:

$$\begin{aligned} \frac{D_a v_b^i}{D\tau} &= \frac{D_b f_b^i}{Dt} - (f_b^k - f_a^k) \nabla_k v_b^i, \\ \frac{D_a \rho_b}{D\tau} &= -\rho_b \nabla_k f_b^k - (f_b^k - f_a^k) \nabla_k \rho_b, \\ \frac{D_a}{D\tau} \nabla_p u_{qb} &= A_{qr} \nabla_p f_r^r - (f_b^k - f_a^k) \nabla_k \nabla_p u_{qb}. \end{aligned} \quad (4.18)$$

Hence, the integrand of Eq. (4.16) becomes

$$\begin{aligned} \sum_a \rho_a \frac{D_a L}{D\tau} &= \sum_a \sum_b \rho_a \left[ L_{v_b^i} \left( \frac{D_b f_b^i}{Dt} - (f_b^k - f_a^k) \nabla_k v_b^i \right) - L_{\rho_b} (\rho_b \nabla_k f_b^k + (f_b^k - f_a^k) \nabla_k \rho_b) \right. \\ &\quad \left. + L_{\nabla_p u_{qb}} (A_{qr} \nabla_p f_b^r - (f_b^k - f_a^k) \nabla_k \nabla_p u_{qb}) \right]. \end{aligned} \quad (4.19)$$

The individual terms can be simplified, yielding sequentially

$$\begin{aligned} \sum_a \sum_b \rho_a L_{v_b^i} \frac{D_b f_b^i}{Dt} &= \sum_b \rho L_{v_b^i} \frac{D_b f_b^i}{Dt} = \sum_b \rho_b \frac{\rho}{\rho_b} L_{v_b^i} \frac{D_b f_b^i}{Dt} \\ &= \sum_b \rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} f_b^i \right) - \sum_b \rho_b f_b^i \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} \right), \\ \sum_a \sum_b \rho_a L_{\rho_b} \rho_b \nabla_k f_b^k &= \sum_b [\nabla_k (\rho \rho_b L_{\rho_b} \nabla_k f_b^k) - f_b^k \nabla_k (\rho \rho_b L_{\rho_b})], \\ \sum_a \sum_b \rho_a L_{\nabla_p u_{qb}} A_{qr} \nabla_p f_b^r &= \sum_b [\nabla_p (\rho L_{\nabla_p u_{qb}} A_{qr} f_b^r) - f_b^r \nabla_p (\rho L_{\nabla_p u_{qb}} A_{qr})], \\ \sum_a \sum_b \rho_a Z_{kb} (f_b^k - f_a^k) &= \sum_b \rho Z_{kb} (f_b^k - F^k), \end{aligned} \quad (4.20)$$

where  $Z_{kb}$  are arbitrary, and

$$F^i = \frac{1}{\rho} \sum_a \rho_a f_a^i,$$

is the “mean” virtual velocity. Combining (4.19) and (4.20) we get the relationship

$$\begin{aligned} \sum_a \rho_a \frac{D_a L}{D\tau} &= \sum_b \left[ \rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} f_b^i \right) - \rho_b f_b^i \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} \right) - \rho L_{v_b^i} \nabla_k v_b^i (f_b^k - F^k) \right. \\ &\quad \left. + f_b^k \nabla_k (\rho \rho_b L_{\rho_b}) - \rho L_{\nabla_p u_{qb}} \nabla_k \nabla_p u_{qb} (f_b^k - F^k) - \rho L_{\rho_b} \nabla_k \rho_b (f_b^k - F^k) \right. \\ &\quad \left. - f_b^k \nabla_p (\rho L_{\nabla_p u_{qb}} A_{qr}) + \nabla_p (\rho L_{\nabla_p u_{qb}} A_{qr} f_b^r) - \nabla_k (\rho \rho_b L_{\rho_b} f_b^k) \right]. \end{aligned} \quad (4.21)$$

Now, let us insert Eq. (4.21) in (4.16) and eliminate all spatial divergence terms - *viz.* the final two in Eq. (4.21) (here and in the following we tacitly assume that all integrals over boundaries vanish: the reader can easily figure out several conservative boundary conditions of this sort), yielding

$$\frac{dS}{d\tau} = \int_{t_0}^{t_1} dt \int_{\Omega} d\Omega \sum_b \left[ \rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} f_b^i \right) - \rho_b f_b^i \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} \right) \right]$$

$$\begin{aligned}
& -\rho L_{v_b^i} \nabla_k v_b^i (f_b^k - F^k) + f_b^k \nabla_k (\rho \rho_b L_{\rho_b}) - \rho L_{\rho_b} \nabla_k \rho_b (f_b^k - F^k) \\
& - f_b^r \nabla_p (\rho L_{\nabla_p u_{qb}} A_{qrb}) - \rho L_{\nabla_p u_{qb}} \nabla_k \nabla_p u_{qb} (f_b^k - F^k) \Big].
\end{aligned} \tag{4.22}$$

Using the second of the two relationships (4.15) we get

$$\int_{t_0}^{t_1} dt \int_{\Omega} d\Omega \sum_b \rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} f_b^i \right) = \sum_b \int_{\Omega} d\Omega \rho L_{v_b^i} f_b^i \Big|_{t=t_1}^{t=t_0}. \tag{4.23}$$

The last substitution vanishes because the initial and final configurations of all admissible trajectories are the same. In view of (4.23) the first integral in Eq. (4.22) disappears. Now, by separating independent virtual velocities  $f_b^i$  in (4.22) and setting  $dS/d\tau = 0$ , we obtain the following governing system of evolution:

$$\begin{aligned}
\rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} \right) &= \rho_b \sum_a \left( L_{v_a^i} \nabla_i v_a^j + L_{\rho_a} \nabla_i \rho_a \right) - \rho L_{v_b^i} \nabla_i v_b^j - \rho L_{\rho_b} \nabla_i \rho_b - \rho L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} \\
&+ \nabla_i (\rho \rho_b L_{\rho_b}) - \nabla_j (\rho L_{\nabla_j u_{kb}} A_{kib}) + \rho_b \sum_a L_{\nabla_j u_{ka}} \nabla_i \nabla_j u_{ka},
\end{aligned}$$

which we can rewrite as

$$\begin{aligned}
\rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} \right) &= \rho_b \nabla_i L - \rho \left( L_{v_b^i} \nabla_i v_b^j + L_{\rho_b} \nabla_i \rho_b + L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} \right) \\
&+ \nabla_i (\rho \rho_b L_{\rho_b}) - \nabla_j (\rho L_{\nabla_j u_{kb}} A_{kib}).
\end{aligned} \tag{4.24}$$

This is the first significant result of the Chapter. In the next Section we will generalize this to include dissipation.

## 4.4 The Hamilton principle for irreversible processes in two-component media

### 4.4.1 The governing equations

Let us consider irreversible processes in two component media. In order to include into consideration the Darcy and Navier-Stokes dissipation we assume the presence of the viscous mass force  $P_{a.vis}^i$  and of the viscous stress tensor  $P_{a.vis}^{ji}$ . Also we extend the Hamilton principle and formulate it in the following nonholonomic form

$$\frac{d}{d\tau} \int_{t_0}^{t_1} dt \int_{\Omega} d\Omega \rho L + \int_{t_0}^{t_1} dt \int_{\Omega} d\Omega \sum_a \left( P_{a.vis}^i \frac{D_a u_{ia}}{D\tau} - P_{a.vis}^{ji} \nabla_j \frac{D_a u_{ia}}{D\tau} \right) = 0. \tag{4.25}$$

Copying the derivation of the system (4.24) we arrive at the following generalization of the governing system

$$\begin{aligned} \rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} \right) &= \rho_b \nabla_i L - \rho \left( L_{v_b^i} \nabla_i v_b^j + L_{\rho_b} \nabla_i \rho_b + L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} \right) \\ &\quad + \nabla_i (\rho \rho_b L_{\rho_b}) - \nabla_j (\rho L_{\nabla_j u_{kb}} A_{kib}) + P_{b.vis}^i + \nabla_j P_{b.vis}^{ji}. \end{aligned} \quad (4.26)$$

This reduces to Eq. (4.24) for conservative systems.

#### 4.4.2 Energy relations and general properties

For each constituent,  $a$ , define the elastic stress tensor

$$P_{.ia}^j = -\rho L_{\nabla_j u_{ka}} A_{kia} + \rho \rho_a L_{\rho_a} \delta_i^j, \quad (4.27)$$

and the “interaction” force, or momentum supply in the terminology of mixture theory [41, 42],

$$G_a^i = \rho_a \nabla_i L - \rho \left( L_{v_a^i} \nabla_i v_a^j + L_{\rho_a} \nabla_i \rho_a + L_{\nabla_j u_{ka}} \nabla_i \nabla_j u_{ka} \right), \quad (4.28)$$

then the equilibrium conditions (4.26) become

$$\rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} \right) = \nabla_j \left( P_b^{ji} + P_{b.vis}^{ji} \right) + G_b^i + P_{b.vis}^i. \quad (4.29)$$

Note that the sum of the interaction forces at a point is zero,

$$\sum_a G_a^i = 0. \quad (4.30)$$

The sum of the elastic and viscous stresses,  $P_a^{ji} + P_{a.vis}^{ji}$ , can be considered the total stress in phase  $a$ . We caution that these definitions are not unique because the vectors  $G_a^i$  could be changed by adding divergence-like terms.

Define the total energy density per unit mass of the system,

$$\mathcal{E} = \sum_b \frac{\partial L}{\partial v_b^k} v_b^k - L, \quad (4.31)$$

and define the flux vector for component  $a$

$$\mathcal{F}_a^j = -v_a^i P_{.ia}^j + \rho (v_a^j - V^j) v_a^k L_{v_a^k}, \quad (4.32)$$

where

$$V^l = \frac{1}{\rho} \sum_b \rho_b v_b^l, \quad (4.33)$$

is the mean velocity. Then it follows from Eq. (4.26), with the details in the Appendix, that

$$\sum_a \left( \rho_a \frac{D_a \mathcal{E}}{Dt} + \nabla_j \mathcal{F}_a^j \right) = \sum_a \left( P_{a.vis}^i v_{ia} - P_{a.vis}^{ji} \nabla_j v_{ia} \right). \quad (4.34)$$

This is the pointwise energy dissipation relation. It is interesting to note that the flux vector of phase  $a$  contains a term proportional to the relative velocity of  $a$  with respect to the mean flow.

Integrating the balance law (4.34) over the domain  $\Omega$  and using Eq. (4.15) leads us to the following identity

$$\frac{d}{dt} \int_{\Omega} d\Omega \rho \mathcal{E} = \int_{\Omega} d\Omega \sum_a \left( P_{a.vi_s}^i v_{ia} - P_{a.vi_s}^{ji} \nabla_j v_{ia} \right), \quad (4.35)$$

which is the energy-dissipation relation for the volume  $\Omega$ . In the absence of dissipation we can establish the energy conservation equation

$$\frac{d}{dt} \int_{\Omega} d\Omega \rho \mathcal{E} = \frac{d}{dt} \int_{\Omega} d\Omega \rho \left( \sum_b \frac{\partial L}{\partial v_b^k} v_b^k - L \right) = 0. \quad (4.36)$$

Alternatively, let  $\mathcal{L}$  be the Lagrangian density per unit volume,

$$\mathcal{L} = \rho L, \quad (4.37)$$

in terms of which the governing equations (4.26) become

$$\begin{aligned} \rho_b \frac{D_b}{Dt} \left( \frac{\mathcal{L}_{v_b^i}}{\rho_b} \right) = & \rho_b \nabla_i \mathcal{L}_{\rho_b} - \mathcal{L}_{v_b^j} \nabla_i v_b^j - \mathcal{L}_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} - \nabla_j (\mathcal{L}_{\nabla_j u_{kb}} A_{kib}) \\ & + P_{b.vi_s}^i + \nabla_j P_{b.vi_s}^{ji}. \end{aligned} \quad (4.38)$$

This form of the equilibrium equations is perhaps simpler. It is interesting to note that if  $\mathcal{L}$  is additive in the following sense,

$$\mathcal{L} = \sum_a \rho_a L_a(\rho_a, v_a^i, \nabla_j u_{ka}), \quad (4.39)$$

then the equations (4.38) reduce to

$$\rho_b \frac{D_b}{Dt} L_{b,v_b^i} = \nabla_j \left( \rho_b^2 L_{b,\rho_b} \delta^{ij} - \rho_b L_{b,\nabla_j u_{kb}} A_{kib} \right) + P_{b.vi_s}^i + \nabla_j P_{b.vi_s}^{ji}. \quad (4.40)$$

These equations are coupled only through the viscous terms. Thus, an additive volumetric Lagrangian density leads to decoupling.

#### 4.4.3 Dissipation effects: Constitutive relations

For the stresses  $P_{b.vi_s}^{ji}$  we limit ourselves with the traditional choice of the rheology

$$P_{b.vi_s}^{ji} = D_b^{ijkl} \nabla_{(k} v_{l)b}, \quad (4.41)$$

where the tensor of viscous coefficients  $D_b^{ijkl}$  is symmetric with respect to the first and second pairs of indices and to interchange of these pairs. Also, based on Eq. (4.34), it should satisfy the following dissipation inequality

$$D_b^{ijkl} \nabla_{(i} v_{j)b} \nabla_{(k} v_{l)b} > 0. \quad (4.42)$$

In order to establish a constitutive relation for the viscous body force  $P_{b.vis}^i$  we impose the following “natural” demands:

a) dissipativity

$$\sum_b P_{b.vis}^i v_{ib} < 0, \quad (4.43)$$

b) additivity

$$P_{b.vis}^i = \sum_a F_{ab}^i, \quad (4.44)$$

c) the “action-reaction” equality

$$F_{ab}^i = -F_{ba}^i, \quad (4.45)$$

d) linearity in the velocities

$$F_{ab}^i = \sum_c D_{abc}^{ij} v_{jc}. \quad (4.46)$$

The constraint a) is a natural consequence of the energy identity (4.34), while the conditions b) and c) imply that

$$\sum_b P_{b.vis}^i = 0, \quad (4.47)$$

which is similar to the general result (4.30).

We now examine the implications of these constraints for two-component media. First, Eq. (4.45) implies that

$$F_{11}^i = F_{22}^i = 0, \quad F_{12}^i = -F_{21}^i. \quad (4.48)$$

Combining (4.46) and (4.48) we get

$$D_{111}^{ij} v_{j1} + D_{112}^{ij} v_{j2} = 0, \quad D_{221}^{ij} v_{j1} + D_{222}^{ij} v_{j2} = 0, \\ (D_{121}^{ij} + D_{211}^{ij}) v_{j1} + (D_{122}^{ij} + D_{212}^{ij}) v_{j2} = 0. \quad (4.49)$$

Equations (4.49) lead us to the following formulas of the coefficients  $D_{abc}^{ij}$ :

$$D_{111}^{ij} = D_{112}^{ij} = 0, \quad D_{221}^{ij} = D_{222}^{ij} = 0$$

$$D_{121}^{ij} = -D_{211}^{ij} = \chi_1^{ij}, \quad D_{122}^{ij} = -D_{212}^{ij} = \chi_2^{ij}. \quad (4.50)$$

Combining Eqs. (4.48) - (4.50) we obtain the relationships

$$F_{12}^i = -F_{21}^i = D_{121}^{ij}v_{j1} + D_{122}^{ij}v_{j2} = \chi_1^{ij}v_{j1} + \chi_2^{ij}v_{j2},$$

$$P_{1.vi_s}^i = F_{21}^i, \quad P_{2.vi_s}^i = F_{12}^i = -F_{21}^i = -P_{1.vi_s}^i.$$

Hence,

$$P_{1.vi_s}^i v_{i1} + P_{2.vi_s}^i v_{i2} = F_{21}^i (v_{i1} - v_{i2}) = -(\chi_1^{ij}v_{j1} + \chi_2^{ij}v_{j2})(v_{i1} - v_{i2}).$$

The inequality (4.43) will be guaranteed provided  $\chi_1^{ij} = -\chi_2^{ij} = D^{ij}$  a positively definite matrix. Indeed, in this case we get

$$P_{1.vi_s}^i v_{i1} + P_{2.vi_s}^i v_{i2} = -D^{ij}(v_{i1} - v_{i2})(v_{j1} - v_{j2}) < 0.$$

With the above mentioned choice we obtain the following constitutive equation:

$$P_{1.vi_s}^i = -D^{ij}(v_{i1} - v_{i2}), \quad P_{2.vi_s}^i = -D^{ij}(v_{i2} - v_{i1}), \quad P_{a.vi_s}^i = -D^{ij}(v_{ia} - v_{ib}). \quad (4.51)$$

Inserting the constitutive relations (4.41) and (4.51) in (4.26) we arrive at the following master system:

$$\begin{aligned} \rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} \right) &= \rho_b \nabla_i L - \rho \left( L_{v_b^j} \nabla_i v_b^j + L_{\rho_b} \nabla_i \rho_b + L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} \right) + \nabla_i (\rho \rho_b L_{\rho_b}) \\ &\quad - \nabla_j (\rho L_{\nabla_j u_{kb}} A_{kib}) - D^{ij}(v_{ib} - v_{ia}) + \nabla_j \left( D_{b.vi_s}^{ijkl} \nabla_k v_{lb} \right), \end{aligned} \quad (4.52)$$

or equivalently,

$$\rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} L_{v_b^i} \right) = \nabla_j P_b^{ji} + G_b^i - D^{ij}(v_{ib} - v_{ia}) + \nabla_j \left( D_{b.vi_s}^{ijkl} \nabla_k v_{lb} \right), \quad (4.53)$$

where the interaction force is

$$\begin{aligned} G_b^i = -G_a^i &= \rho_b \left( L_{v_a^j} \nabla_i v_a^j + L_{\rho_a} \nabla_i \rho_a + L_{\nabla_j u_{ka}} \nabla_i \nabla_j u_{ka} \right) \\ &\quad - \rho_a \left( L_{v_b^j} \nabla_i v_b^j + L_{\rho_b} \nabla_i \rho_b + L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} \right). \end{aligned} \quad (4.54)$$

## 4.5 Examples

### 4.5.1 One-component models.

**Example 1. Nonlinear elastic solid.**

The governing system of a nonlinear one-component solid can be derived from the master system (4.53) if we neglect viscous forces and choose the Lagrangian

$$L = T - E = \frac{1}{2}v^i v_i - E(\nabla_j u_k). \quad (4.55)$$

In this case  $P^{ji}$  of (4.27) is the Cauchy stress tensor

$$P^{ji} = \rho \frac{\partial E}{\partial \nabla_j u_k} A_k^i,$$

and there are no viscous or interaction terms. The equilibrium equations (4.53) are simply

$$\rho \frac{Dv^i}{Dt} = \nabla_j P^{ji}. \quad (4.56)$$

### Example 2. The Navier-Stokes fluid.

The governing system of a nonlinear one-component viscous fluid can be derived from the master system (4.52) if we ignore the drag force  $P_{vis}^i$  and choose the Lagrangian and the viscosity tensor  $D^{ijkl}$  in the following form:

$$L = T - E = \frac{1}{2}v^i v_i - E(\rho), \quad D^{ijkl} = \nu_v \delta^{ij} \delta^{kl} + \nu_s (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}), \quad (4.57)$$

where  $\nu_v, \nu_s$  are the volume and shear viscosities, respectively. The elastic stress is hydrostatic, defined by a pressure  $p$ ,

$$P^{ji} = -p \delta^{ji}, \quad p = \rho^2 \frac{\partial E}{\partial \rho}, \quad (4.58)$$

and the viscous stresses  $P_{vis}^{ij}$  are

$$P_{vis}^{ij} = \nu_v \nabla^k v_k \delta^{ij} + \nu_s (\nabla^i v^j + \nabla^j v^i). \quad (4.59)$$

The equilibrium equations (4.53) are now

$$\rho \frac{Dv^i}{Dt} = -\nabla^i p + \nabla_j P_{vis}^{ij}. \quad (4.60)$$

As always, when combined with the mass conservation equation (4.3) the system (4.59), (4.60) is closed in terms of  $\rho, v^i$ .

#### 4.5.2 Two-component models.

Let us assume that the kinetic energy  $T$  is a quadratic form of the velocities  $v_a^k$

$$T = \frac{1}{2} \sum_{a,b} K_{ijab} v_a^i v_b^j. \quad (4.61)$$

Using (4.61) we get

$$\frac{\partial T}{\partial v_c^l} = \frac{1}{2} \sum_b K_{ijab} (\delta_l^i \delta_{ac} v_b^j + v_a^i \delta_l^j \delta_{bc} v_b^j) = \frac{1}{2} \sum_b (K_{ljac} + K_{jlbc}) v_b^j. \quad (4.62)$$

In the case of isotropic two-phase medium we get

$$K_{ab}^{ij} = \tilde{k}_{ab} \delta^{ij}, \quad D^{ij} = \Delta \delta^{ij}. \quad (4.63)$$

Combining (4.62) and (4.63) we get

$$\frac{\partial T}{\partial v_c^l} = \sum_a k_{ac} v_{la}, \quad (4.64)$$

where the symmetric matrix  $k_{ac}$  is defined as

$$k_{ac} = \frac{1}{2} (\tilde{k}_{ac} + \tilde{k}_{ca}).$$

### Example 3. Liquid two-component mixture.

We obtain the simplest possible model of a two-component liquid mixture by choosing the Lagrangian and viscosity tensors in the following form:

$$\begin{aligned} L &= T - E = \frac{1}{2} \sum_{ab} k_{ab} v_a^i v_{ib} - E(\rho_a), \\ D^{ij} &= \Delta \delta^{ij}, \quad D_a^{ijkl} = \nu_{va} \delta^{ij} \delta^{kl} + \nu_{sa} (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}). \end{aligned} \quad (4.65)$$

The partial stresses are hydrostatic,

$$P_b^{ji} = -p_b \delta^{ji}, \quad p_b = \rho \rho_b \frac{\partial E}{\partial \rho_b}, \quad (4.66)$$

and the interaction forces are given by

$$\begin{aligned} G_b^i &= \rho_b \nabla_i \left( \frac{1}{2} \sum_{ac} k_{ac} v_a^i v_{ic} - E \right) - \rho \sum_a k_{ab} v_{ja} \nabla_i v_b^j + \rho E_{\rho_b} \nabla_i \rho_b \\ &= \sum_c v_{ic} (\rho_b k_{ca} \nabla_i v_a^j - \rho_a k_{cb} \nabla_i v_b^j) + \rho_a E_{\rho_b} \nabla_i \rho_b - \rho_b E_{\rho_a} \nabla_i \rho_a. \end{aligned} \quad (4.67)$$

Inserting these into the master equations (4.53) we obtain the following system of equilibrium equations for the two-liquid mixture

$$\begin{aligned} \rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} \sum_a k_{ab} v_{ia} \right) &= \sum_c v_{ic} (\rho_b k_{ca} \nabla_i v_a^j - \rho_a k_{cb} \nabla_i v_b^j) + \rho_a E_{\rho_b} \nabla_i \rho_b - \rho_b E_{\rho_a} \nabla_i \rho_a \\ &\quad - \nabla_i (\rho \rho_b E_{\rho_b}) - \Delta (v_{ib} - v_{ia}) + \nabla_j [\nu_{vb} \delta_i^j \nabla_k v_b^k + \nu_{sb} (\nabla_i v_b^j + \nabla^j v_{ib})]. \end{aligned} \quad (4.68)$$

**Example 4. Solid two-component mixture.**

The simplest model of a two-component solid mixture corresponds to a Lagrangian of the form:

$$L = T - E = \frac{1}{2} \sum_{ab} k_{ab} v_a^i v_{ib} - E(\nabla_i u_{ja}). \quad (4.69)$$

Inserting (4.69) in (4.53) we obtain the following master system for a two-component solid elastic mixture:

$$\begin{aligned} \rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} \sum_a k_{ab} v_{ia} \right) - \rho_b \nabla_i \left( \frac{1}{2} \sum_{ac} k_{ac} v_a^i v_{ic} \right) + \rho \sum_a k_{ab} v_{jb} \nabla_i v_b^j &= -\rho_b \nabla_i E \\ + \rho E_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} + \nabla_j (\rho E_{\nabla_j u_{kb}} A_{kib}) - D^{ij} (v_{ib} - v_{ia}) + \nabla_j \left( D_{b,vis}^{ijkl} \nabla_k v_{la} \right). \end{aligned} \quad (4.70)$$

**Example 5. Fluid permeable elastic solid.**

We choose subscript *s* for the solid and *f* for the liquid component. The simplest model of two-component solid mixture is obtained by choosing the Lagrangian in the following form:

$$L = T - E = \frac{1}{2} \left( k_{ss} v_s^i v_{is} + 2k_{sf} v_s^i v_{if} + k_{ff} v_f^i v_{if} \right) - E(\nabla_i u_{js}, \rho_f). \quad (4.71)$$

Inserting (4.71) in (4.53) yields the following master system:

$$\begin{aligned} \rho_s \frac{D_s}{Dt} \left( \frac{\rho}{\rho_s} (k_{ss} v_{is} + k_{sf} v_{if}) \right) &= \rho_s (k_{fs} v_{js} + k_{ff} v_{jf}) \nabla_i v_f^j - \rho_f (k_{ss} v_{js} + k_{sf} v_{jf}) \nabla_i v_s^j \\ - \rho_s \nabla_i E + \rho E_{\nabla_j u_{ks}} \nabla_i \nabla_j u_{ks} + \nabla_j (\rho E_{\nabla_j u_{ks}} A_{kis}) & \\ - D^{ij} (v_{js} - v_{jf}) + \nabla_j \left( D_{s,vis}^{ijkl} \nabla_k v_{ls} \right), \end{aligned} \quad (4.72)$$

$$\begin{aligned} \rho_f \frac{D_f}{Dt} \left( \frac{\rho}{\rho_f} (k_{ff} v_{if} + k_{sf} v_{is}) \right) &= \rho_f (k_{ss} v_{js} + k_{sf} v_{jf}) \nabla_i v_s^j - \rho_s (k_{fs} v_{js} + k_{ff} v_{jf}) \nabla_i v_f^j \\ - \rho_f \nabla_i E + \rho E_{\rho_f} \nabla_i \rho_f - \nabla_i (\rho \rho_f E_{\rho_f}) & \\ - D^{ij} (v_{jf} - v_{js}) + \nabla_j \left( D_{f,vis}^{ijkl} \nabla_k v_{lf} \right). \end{aligned} \quad (4.73)$$

## 4.6 Conclusion

We have established a general framework for dealing with fully nonlinear problems for two-component media. The central result is Eq. (4.26) which gives the equilibrium equations in Eulerian coordinates for an arbitrary Lagrangian density. These governing equations are a direct

consequence of the generalized Hamilton's principle in the presence of dissipative effects, *i.e.*, the Onsager-Sedov principle of Eq. (4.25). We have illustrated the application of the general governing equations to specific examples of single and two-phase continua. In particular, the standard theories of solids and fluids are included. The full potential of the theory is its applicability to multi-phase continua.

## 4.7 Appendix: Some details

Here we derive the energy identity (4.34). We start with

$$\begin{aligned} \sum_a \rho_a \frac{D_a \mathcal{E}}{Dt} &= \sum_a \rho_a \frac{D_a}{Dt} \left( \sum_b \frac{\partial L}{\partial v_b^k} v_b^k - L \right) \\ &= \sum_a \rho_a \left( \sum_b v_b^k \frac{D_a}{Dt} \left( \frac{\partial L}{\partial v_b^k} \right) - \sum_b L_{\rho} \frac{D_a \rho_b}{Dt} - \sum_b L_{\nabla_j u_{lb}} \frac{D_a}{Dt} \nabla_j u_{lb} \right). \end{aligned} \quad (\text{A.1})$$

We next simplify each of these terms in turn. First, using (4.17),

$$\begin{aligned} \sum_b v_b^k \frac{D_a}{Dt} \left( \frac{\partial L}{\partial v_b^k} \right) &= \sum_b v_b^k \frac{D_b}{Dt} \frac{\partial L}{\partial v_b^k} - \sum_b v_b^k (v_b^l - v_a^l) \nabla_l \frac{\partial L}{\partial v_b^k} \\ &= \sum_b v_b^k \left[ \frac{\rho_b}{\rho} \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} \frac{\partial L}{\partial v_b^k} \right) - \frac{\partial L}{\partial v_b^k} \frac{\rho_b}{\rho} \frac{D_b}{Dt} \frac{\rho}{\rho_b} - (v_b^l - v_a^l) \nabla_l \frac{\partial L}{\partial v_b^k} \right], \end{aligned}$$

which implies that

$$\sum_a \rho_a \sum_b v_b^k \frac{D_a}{Dt} \left( \frac{\partial L}{\partial v_b^k} \right) = \sum_b v_b^k \rho_b \left[ \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} \frac{\partial L}{\partial v_b^k} \right) - \frac{\partial L}{\partial v_b^k} \frac{D_b}{Dt} \frac{\rho}{\rho_b} \right] - \rho \sum_b v_b^k (v_b^l - V^l) \nabla_l \frac{\partial L}{\partial v_b^k}, \quad (\text{A.2})$$

and  $V^l$  is defined in (4.33). By making use of the governing system (4.26) we get the following result:

$$\begin{aligned} \sum_b v_b^k \rho_b \frac{D_b}{Dt} \left( \frac{\rho}{\rho_b} \frac{\partial L}{\partial v_b^k} \right) &= \sum_b v_b^i \left[ \rho_b \nabla_i L - \rho \left( L_{v_b^k} \nabla_i v_b^k + L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} + L_{\rho_b} \nabla_i \rho_b \right) \right. \\ &\quad \left. - \nabla_j \left( \rho L_{\nabla_j u_{kb}} A_{kib} \right) + \nabla_i (\rho \rho_b L_{\rho_b}) + P_{vis,b}^i + \nabla_j P_{vis,b}^{ji} \right]. \end{aligned} \quad (\text{A.3})$$

Use of Eq. (4.3) gives us

$$\begin{aligned} \rho_b \frac{D_b}{Dt} \frac{\rho}{\rho_b} &= \sum_a \frac{D_b \rho_a}{Dt} - \frac{\rho}{\rho_b} \frac{D_b \rho_b}{Dt} = \sum_a \left[ -\rho_a \nabla_l v_a^l - (v_a^l - v_b^l) \nabla_l \rho_a + \rho_a \nabla_l v_b^l \right] \\ &= \nabla_l \left( \rho (v_b^l - V^l) \right), \end{aligned}$$

implying

$$\sum_b v_b^k \rho_b \frac{\partial L}{\partial v_b^k} \frac{D_b}{Dt} \frac{\rho}{\rho_b} = \sum_b v_b^k \frac{\partial L}{\partial v_b^k} \nabla_l \left( \rho (v_b^l - V^l) \right). \quad (\text{A.4})$$

Combining Eqs. (A.2), (A.3), and (A.4) gives us the first sum in the left member of Eq. (A.1),

$$\begin{aligned} \sum_a \sum_b \rho_a v_b^k \frac{D_a}{Dt} \left( \frac{\partial L}{\partial v_b^k} \right) &= \rho V^i \nabla_i L + \sum_b v_b^i \left[ -\rho \left( L_{v_b^k} \nabla_i v_b^k + L_{\nabla_j u_{kb}} \nabla_i \nabla_j u_{kb} + L_{\rho_b} \nabla_i \rho_b \right) \right. \\ &\quad \left. - \nabla_j \left( \rho L_{\nabla_j u_{kb}} A_{kib} \right) + \nabla_i (\rho \rho_b L_{\rho_b}) + P_{vis.b}^i + \nabla_j P_{vis.b}^{ji} \right. \\ &\quad \left. + \nabla_l \left( \rho L_{v_b^i} (V^l - v_b^l) \right) \right]. \end{aligned} \quad (\text{A.5})$$

Now consider the other terms in (A.1). Using the mass conservation Eq. (4.3) again gives

$$\begin{aligned} \sum_a \sum_b \rho_a L_{\rho_b} \frac{D_a \rho_b}{Dt} &= \sum_a \sum_b \rho_a L_{\rho_b} \left[ -\rho_b \nabla_k v_b^k - (v_b^k - v_a^k) \nabla_k \rho_b \right] \\ &= \sum_b \rho L_{\rho_b} \left[ V^k \nabla_k \rho_b - \nabla_k (\rho_b v_b^k) \right]. \end{aligned} \quad (\text{A.6})$$

Also, by using of the last of the Eq. (4.18) (with the change of the parameter  $\tau$  for  $t$ ) we get the following equation:

$$\begin{aligned} \sum_a \sum_b \rho_a L_{\nabla_j u_{lb}} \frac{D_a}{Dt} \nabla_j u_{lb} &= \sum_a \sum_b \rho_a L_{\nabla_j u_{lb}} \left[ A_{lrb} \nabla_j v_b^r - (v_b^k - v_a^k) \nabla_k \nabla_j u_{lb} \right] \\ &= \sum_b \rho L_{\nabla_j u_{lb}} \left[ A_{lrb} \nabla_j v_b^r - v_b^k \nabla_k \nabla_j u_{lb} + V^k \nabla_k \nabla_j u_{lb} \right]. \end{aligned} \quad (\text{A.7})$$

By combining (A.1), (A.5), (A.6), and (A.7) we obtain the relation (4.34), where the flux vectors are defined in (4.32).

## Chapter 5

# Acoustoelasticity of fluid-saturated porous media

The general theory for small dynamic motion superimposed upon large static deformation, or acoustoelasticity, is developed for isotropic fluid-filled poroelastic solids. Formulae are obtained for the change in acoustic wave speeds for arbitrary loading, both on the frame and the pore fluid. Specific experiments are proposed to find the complete set of third order elastic moduli for an isotropic poroelastic medium. Because of the larger number of third order moduli involved, 7 as compared with 3 for a simple elastic medium, experiments combining open-pore, closed-pore, jacketed, and unjacketed configurations are required. The details for each type of loading are presented, and a set of possible experiments is discussed. The present theory is applicable to fluid-saturated, bi-connected porous solids, such as sandstones or consolidated granular media.

### 5.1 Introduction

The linear theory of poroelastic fluid saturated media is a mature subject, with the classic studies of Frenkel [6] and especially of Biot [7] standing as the significant achievements in its modern development. Several different methods now exist for arriving at governing equations, including the theory of interpenetrating continua [8], and the method of “homogenization” based on two-scale asymptotic expansions [59, 10]. Despite the fact that these approaches do not always yield the same governing equations, and further general theoretical studies are required, they have certainly proved useful in several branches of civil and geo-engineering and the Earth sciences [11]-[13], and also in unexpected fields, such as low temperature physics [14] and pattern formation in polymer gels [15]. However, there are definite limitations to the linear theory. For example, a proper analysis of the dynamics of large amplitude sound in sediments, or of small amplitude sound in rocks under large confining stress are obviously impossible within the framework of a linear theory. Even seismic waves of very small intensity at a large distance from their epicenter should be studied in the framework of a nonlinear theory because of the well-known effect of the accumulation of nonlinear distortions leading to the “gradient catastrophe” phenomenon [16]-[17].

There has been renewed interest in the nonlinear acoustics of rocks, based partly on several observations of distinct nonlinear effects. These include, the change of the *in situ* velocities of seismic waves [18]-[20], and direct measurement of harmonic distortion [21, 22]. These experiments have clearly demonstrated that the *in situ* nonlinear elastic moduli of rocks, soil and sediments are much greater than the linear ones. The nonlinear behavior of fluid-filled poroelastic solids is not as well understood, and further progress depends upon accurate experimental measurement of the effective nonlinear moduli of poroelastic media.

The determination of the linear moduli has been the subject of numerous experimental and theoretical studies [23, 24, 12]. An isotropic poroelastic medium has four static moduli, which may be measured by a combination of stress-strain experiments on "jacketed" and "unjacketed" samples. The role of the jacketing is to constrain either the fluid pressure or its mass. The nonlinear moduli, in contrast, should be determined by measurement of essentially nonlinear effects. One of the simplest nonlinear phenomena is the elastoacoustic effect, whereby the speeds of small amplitude waves are changed by applying stress or strain. The theory of acoustoelasticity has been thoroughly discussed for purely elastic materials [25]-[27], and is now commonly used in ultrasonics [28]-[30], and has been adapted to multiphase materials [5].

The subject of this Chapter is the acoustoelastic effect in the context of poroelasticity. The theory is based on a nonlinear generalization of the classic Biot theory. Biot himself discussed thermodynamic aspects of a nonlinear theory of poroelastic media, but did not specify any nonlinear stress-strain behavior (see Papers 15, 16, 18 and 19 in ref. [7]). There has been some work on nonlinear poroelasticity within the framework of the theory of interpenetrating continua [8], and also using the method of two-scale homogenization [31, 2]. Three sources of nonlinearity are traditionally distinguished in elasticity: (i) the physical nonlinearity, that is the nonlinearity of the constitutive relations relating the stress and the displacement gradient; (ii) nonlinearity of the universal equations, such as the equations of conservation of mass, momentum, *etc.*; (iii) geometric nonlinearity, which results from the nonlinear relationship between the deformation gradient and the tensor of finite deformations.

In order to keep the analysis as simple as possible, in this Chapter we ignore all dissipative effects and concentrate on the first kind of nonlinearity only because it appears to be the most significant for geophysical materials. We begin with a discussion of *acceleration waves* in nonlinear poroelastic media. These are exact solutions independent of the state of prestress in the medium, and they reduce to the well-known Biot fast and slow waves for unstressed isotropic materials. These general results serve as the basis for considering the principal body waves propagating in a slightly prestressed isotropic nonlinear saturated poroelastic substance. Finally, we discuss application of the general formulae for the stress dependence of the wave speeds to several specifically designed experimental configurations.

## 5.2 Nonlinear poroelasticity

### 5.2.1 Governing equations

All further analysis is based on the following governing equations of the nonviscous, fluid-filled poroelastic medium:

$$\rho \frac{\partial^2 u^i}{\partial t^2} + \rho_f \frac{\partial^2 w^i}{\partial t^2} = \tau^{ij}_{,j}, \quad (\text{A.1a})$$

$$\rho_f \frac{\partial^2 w^i}{\partial t^2} + \rho_f K^{ij} \frac{\partial^2 u_j}{\partial t^2} = -K^{ij} p_{,j}, \quad (\text{A.1b})$$

where

$$\tau^{ij} = \frac{\partial W}{\partial u_{i,j}}(\zeta, u_{m,n}), \quad p = \frac{\partial W}{\partial \zeta}(\zeta, u_{m,n}). \quad (\text{A.2})$$

Here  $u^i$  is the displacement of the solid skeleton,  $w^i$  is the relative displacement of the fluid,  $\tau^{ij}$  are the stresses,  $p$  is the fluid pressure,  $W(\zeta, u_{m,n})$  is the poro-elastic potential,  $\rho$  and  $\rho_f$  are the averaged density and the density of the fluid, respectively;  $-\zeta = w^i_{,i}$  is a divergence of the relative displacement of the fluid,  $K^{ij}$  is the “instantaneous” magnitude of a symmetric tensor of permeability (the value of the permeability hereditary operator at  $t = 0$ ). Also,  $x^i$  are the spatial coordinates, the Latin indices take the values 1, 2, and 3; summation over repeated indices is implied, and a comma followed by a Latin suffix symbolizes partial differentiation. We refer to the publications [7], [9], [31], and particularly [2] and for the motivation behind Eqs. (A.1) and (A.2).

By inserting the value of  $\zeta$  in terms of  $w^i$  we can rewrite the equations of motion as

$$\rho \frac{\partial^2 u^i}{\partial t^2} + \rho_f \frac{\partial^2 w^i}{\partial t^2} = W^{ijkl} u_{k,jl} - W_{\zeta}^{ij} w_{,jk}^k, \quad (\text{A.3a})$$

$$\rho_f \frac{\partial^2 w^i}{\partial t^2} + \rho_f K^{ij} \frac{\partial^2 u_j}{\partial t^2} = -K^{ij} (W_{\zeta}^{kl} u_{k,lj} - W_{\zeta\zeta} w_{,kj}^k). \quad (\text{A.3b})$$

In these and subsequent equations we use the following notation for the derivatives of the potential energy function  $W$ :  $W^{ij} = \partial W / \partial u_{i,j}$ ,  $W_{\zeta\zeta} = \partial^2 W / \partial \zeta^2$ ,  $W_{\zeta}^{ij} = \partial^2 W / \partial u_{i,j} \partial \zeta$ ,  $W^{ijkl} = \partial^2 W / \partial u_{i,j} \partial u_{k,l}$ , etc.

### 5.2.2 Acceleration waves

An acceleration wave-front is a propagating surface defined such that displacements and their first derivatives are continuous across this surface, but the second and higher derivatives and, in particular, the accelerations possess finite jumps. Acceleration waves are the most convenient object of theoretical study for both linear and nonlinear dynamics. They have been investigated thoroughly by Hadamard [32] and Thomas [33], while Chen [34] provides a more recent review for elastic materials. Continuity of the first derivatives imposes geometric and kinematic constraints on possible jumps of the second derivatives since the “tangential” components of the second

derivatives are continuous. These constraints are known as compatibility conditions, and are essentially geometric conditions, originally developed by outstanding geometers like Hadamard [32], Levi-Civita [35], and Thomas [33] (fortunately all three were outstanding mathematical physicists and their works are accessible to physicists). In particular, the jumps of second derivatives across acceleration wave-fronts satisfy [34, 36]

$$[u_{i,jk}]^+_- = h_i n_j n_k, \quad \left[ \frac{\partial^2 u_i}{\partial t^2} \right]_-^+ = h_i c^2, \quad (\text{A.4a})$$

$$[w_{i,jk}]^+_- = H_i n_j n_k, \quad \left[ \frac{\partial^2 w_i}{\partial t^2} \right]_-^+ = H_i c^2, \quad (\text{A.4b})$$

where  $h_k = [u_{i,pq}]^+_- n^p n^q$  and  $H_k = [w_{i,pq}]^+_- n^p n^q$  are the second order amplitude vectors of discontinuity;  $n^j$  is the unit normal to the wave-front, and  $c$  is the velocity of the front. The compatibility conditions indicate that the two vector functions  $h_k$  and  $H_k$  completely define the jumps of 60 partial derivatives of the displacements.

In order to determine the acceleration vectors themselves one has to extract some additional dynamic information from the governing Eqs. (A.3). These equations are not valid at the wave-front since the second derivatives are undefined at the front. By definition, only one-sided second derivatives are defined at the front, and these one-sided limits satisfy Eqs. (A.3). We first subtract term-wise Eq. (A.3a) for the two one sided limits taking into account continuity of the first derivatives and, then do the same operation with Eq. (A.3b). Then, evaluating the jumps of the second derivatives using the second order compatibility conditions of Eq. (A.4) we get the following linear algebraic system for the amplitude vectors

$$(\rho c^2 \delta^{ik} - W^{ijkl} n_j n_l) h_k + (\rho_f c^2 \delta^{ik} + W_\zeta^{ij} n_j n^k) H_k = 0, \quad (\text{A.5a})$$

$$(\rho_f c^2 \delta^{ik} + W_\zeta^{kl} n_l n^i) h_k + (\rho_f c^2 K_{inv}^{ik} - W_{\zeta\zeta} n^i n^k) H_k = 0, \quad (\text{A.5b})$$

where  $K_{inv}^{ij}$  is the inverse of the permeability matrix  $K^{ij}$ , which is assumed to be invertible.

The pressure  $p$  is continuous across the wave front, but its gradient is not, specifically

$$[p_i]_-^+ = q n_i. \quad (\text{A.6})$$

At the same time, it follows from the Eqs. (A.2) and (A.4) that

$$[p_i]_-^+ = W_\zeta^{kl} n_i n_l h_k - W_{\zeta\zeta} n_i n^k H_k. \quad (\text{A.7})$$

Combining the previous two equations allows us to eliminate the quantity  $n^k H_k$  in favor of the jump in the pressure gradient,  $q$ . Then Eq. (A.5b) provides an expression for  $H^i$ ,

$$H^i = -K^{ij} \left( h_j + \frac{q}{\rho_f c^2} n_j \right). \quad (\text{A.8})$$

After some manipulation, Eqs. (A.5) imply that

$$(\tilde{\rho}^{ik}c^2 - \tilde{W}^{ijkl}n_j n_l) h_k - \tilde{K}^{ij}n_j q = 0, \quad (\text{A.9a})$$

$$\tilde{K}^{kl}n_l h_k + \left( \frac{K^{ij}n_i n_j}{\rho_f c^2} - \frac{1}{W_{\zeta\zeta}} \right) q = 0, \quad (\text{A.9b})$$

where

$$\tilde{\rho}^{ik} = \rho \delta^{ik} - \rho_f K^{ik}, \quad (\text{A.10a})$$

$$\tilde{W}^{ijkl} = W^{ijkl} - W_{\zeta}^{ij} W_{\zeta}^{kl} / W_{\zeta\zeta}, \quad (\text{A.10b})$$

$$\tilde{K}^{ik} = K^{ik} + W_{\zeta}^{ik} / W_{\zeta\zeta}. \quad (\text{A.10c})$$

Equations (A.9) form a system of 4 linear algebraic equations with respect to the 4 unknowns  $h_k$  and  $q$ . This system has a non-zero solution when the determinant vanishes, leading to a 4-th order polynomial equation in  $c^2$ . The coefficients of the polynomial depend on  $u_{k,l}$ ,  $\zeta$  and the orientation of the unit normal  $n_k$ , and thus, the same is true for the velocities  $c$ . Each real root generates an associated non-zero solution  $(h_k, q)$ . Further discussion of the 4-th order system can be found in [37]. On the other hand, the system defined by Eq. (A.5) is 6-th order, because it involves the unknowns  $h_i$  and  $H_j$ . The 4-th and 6-th order systems are connected by the fact that the latter possesses the root  $c^2 = 0$  of multiplicity 2. Thus, when  $c^2 = 0$  the system of Eq. (A.5) has infinitely many solutions of the form  $h_k = 0$ ,  $H_k = R_k$  where  $R_k$  is an arbitrary vector in the 2D-space orthogonal to the wave normal  $n_k$ . In summary, the propagating waves and the non-zero wave speeds can be found from either the 4-th order or the 6-th order systems.

### 5.2.3 Principal directions

A principal direction is defined as one in which an eigenvector, such as  $h^i$ , is aligned with or orthogonal to the wave normal  $n^i$ . For example, consider a longitudinal wave in the direction of  $n_i$ , that is, a solution of the form

$$h_k = h n_k, \quad H_k = H n_k. \quad (\text{A.11})$$

Substituting from Eq. (A.11) into Eq. (A.5) and contracting with  $n_i$  gives

$$(\rho c^2 - W^{ijkl}n_i n_j n_k n_l) h + (\rho_f c^2 + W_{\zeta}^{il}n_i n_l) H = 0, \quad (\text{A.12a})$$

$$(\rho_f c^2 + W_{\zeta}^{il}n_i n_l) h + (\rho_f c^2 K_{inv}^{ik}n_i n_k - W_{\zeta\zeta}) H = 0. \quad (\text{A.12b})$$

It follows immediately from Eqs. (A.12) that the corresponding eigenvalues  $c^2$  are defined by a biquadratic characteristic equation, whereas the amplitudes  $h$  and  $H$  are connected by the relation

$$\frac{\rho c^2 - W^{ijkl}n_i n_j n_k n_l}{\rho_f c^2 K_{inv}^{ik}n_i n_k - W_{\zeta\zeta}} = \frac{H^2}{h^2}. \quad (\text{A.13})$$

A transverse wave, on the other hand, satisfies  $h_k = h e_k$ ,  $H_k = H e_k$  where  $e^i$  is a polarization vector with  $n^k e_k = 0$  and  $e^k e_k = 1$ . Under these circumstances, Eqs. (A.5) become

$$(\rho c^2 \delta^{ik} - W^{ijkl} n_j n_l) e_k h + \rho_f c^2 e^i H = 0, \quad (\text{A.14a})$$

$$(\rho_f c^2 \delta^{ik} + W_\zeta^{kl} n_l n^i) e_k h + \rho_f c^2 K_{inv}^{ik} e_k H = 0, \quad (\text{A.14b})$$

implying that the transverse wave velocity is given by

$$c_t^2 = \left( \rho - \rho_f (K_{inv}^{pq} e_p e_q)^{-1} \right)^{-1} W^{ijkl} e_i e_k n_j n_l. \quad (\text{A.15})$$

When the material is isotropic then all directions are principal directions, *i.e.*, all acceleration waves are either longitudinal or transverse in nature.

## 5.3 The acoustoelastic effect

### 5.3.1 Third order moduli for poroelasticity

We now consider the effects of prestress on the longitudinal and transverse waves in an isotropic medium. The permeability for an isotropic poroelastic medium is  $K^{ij} = \kappa \delta^{ij}$ , while the potential  $W$  is a function of  $\zeta$  and the principal invariants  $I_M$  of the symmetric strain  $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$ . Thus,

$$W = W(I_1, I_2, I_3, \zeta), \quad (\text{A.16})$$

where

$$I_1 = \delta^{ij} \varepsilon_{ij}, \quad I_2 = \frac{1}{2} (\delta^{qj} \delta^{rk} - \delta^{rj} \delta^{qk}) \varepsilon_{jq} \varepsilon_{kr}, \quad I_3 = \frac{1}{6} \varepsilon^{ijk} \varepsilon^{pqr} \varepsilon_{ip} \varepsilon_{jq} \varepsilon_{kr}, \quad (\text{A.17})$$

and  $\varepsilon^{ijk}$  is the third order alternating tensor. To within third order terms in  $\varepsilon_{ij}$  and  $\zeta$  the potential  $W$  can be approximated by the polynomial

$$\begin{aligned} W = & \Gamma_{11} I_1^2 + \Gamma_2 I_2 + \Gamma_{1\zeta} I_1 \zeta + \Gamma_{\zeta\zeta} \zeta^2 + \Gamma_{111} I_1^3 + \Gamma_{12} I_1 I_2 \\ & + \Gamma_3 I_3 + \Gamma_{1\zeta\zeta} I_1 \zeta^2 + \Gamma_{11\zeta} I_1^2 \zeta + \Gamma_{2\zeta} I_2 \zeta + \Gamma_{\zeta\zeta\zeta} \zeta^3. \end{aligned} \quad (\text{A.18})$$

We consider the following field of perturbations with respect to the reference (unstressed) configuration

$$\delta u_i = \left( \alpha_N n_j x^j n_i + \sum_{A=1,2} \alpha_A x^j e_{Aj} \right) \varepsilon, \quad \delta \zeta = \varepsilon \Delta, \quad (\text{A.19})$$

where  $e_{1i}$  and  $e_{2i}$  are two mutually orthogonal transverse vectors, *i.e.*, both are orthogonal to the  $n^i$  direction. The perturbation magnitude is defined by  $\varepsilon$ , with  $|\varepsilon| \ll 1$  by assumption, and

the parameters  $\alpha_N$ ,  $\alpha_A$ ,  $A = 1, 2$ , and  $\Delta$  are  $O(1)$  and independent. The prestrain is, from Eq. (A.19),

$$\varepsilon_{ij} = \left( \alpha_N n_j n_i + \sum_{A=1,2} \alpha_A e_{Ai} e_{Aj} \right) \varepsilon. \quad (\text{A.20})$$

Differentiating Eq. (A.18) in the vicinity of the reference configuration ( $\varepsilon = 0$ ) we obtain the second order elasticity tensors in terms of the elastic moduli:

$$W^{ijkl} = \lambda_c \delta^{ij} \delta^{kl} + \mu (\delta^{il} \delta^{jk} + \delta^{ik} \delta^{jl}), \quad W_{\zeta}^{ij} = -\alpha M \delta^{ij}, \quad W_{\zeta\zeta} = M, \quad (\text{A.21})$$

where  $\lambda_c$ ,  $\alpha$ , and  $M$  are the Biot parameters [38]. In fact, Biot defines the linearized stress-strain relations in terms of a quadratic energy potential similar to Eq. (A.18). Comparison of the quadratic terms in the latter with equation (3.4) of Biot [38] immediately implies that

$$\lambda_c = 2\Gamma_{11} + \Gamma_2, \quad \mu = -\frac{1}{2} \Gamma_2, \quad M = 2\Gamma_{\zeta\zeta}, \quad \alpha = -\Gamma_{1\zeta}/(2\Gamma_{\zeta\zeta}). \quad (\text{A.22})$$

There are many different notations used for describing the linear response of saturated porous media, and even Biot's own notation evolved from the time of his early work on consolidation (see Paper 1 of ref. [7]) to his later work. His 1962 paper [38] provides a good comparison of the notations used.

For the third order quantities we get

$$\begin{aligned} W_{\zeta}^{ijkl} &= (\Gamma_{2\zeta} + 2\Gamma_{11\zeta}) \delta^{ij} \delta^{kl} - \frac{1}{2} \Gamma_{2\zeta} (\delta^{il} \delta^{jk} + \delta^{ik} \delta^{jl}), \\ W_{\zeta\zeta}^{ij} &= 2\Gamma_{1\zeta\zeta} \delta^{ij}, \quad W_{\zeta\zeta\zeta} = 6\Gamma_{\zeta\zeta\zeta}, \\ W^{ijklmn} &= (6\Gamma_{111} + 3\Gamma_{12}) \delta_{ij} \delta_{kl} \delta_{mn} + \frac{1}{4} \Gamma_3 (\epsilon^{ikm} \epsilon^{jln} + \epsilon^{jkm} \epsilon^{iln} + \epsilon^{ilm} \epsilon^{jkn} + \epsilon^{ilm} \epsilon^{ikn}) \\ &\quad - \frac{1}{2} \Gamma_{12} (\delta^{ij} (\delta^{km} \delta^{ln} + \delta^{kn} \delta^{lm}) + \delta^{kl} (\delta^{im} \delta^{jn} + \delta^{in} \delta^{jm}) + \delta^{mn} (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})). \end{aligned} \quad (\text{A.23})$$

In the absence of poroelastic effects, the third order elastic moduli  $W_{ijklmn}$  are equivalent to the standard third order moduli  $C_{ijklmn}$ , or  $C_{IJK}$  in the concise Voigt notation [28]. In order to compare the moduli here with the more common notation, we note the equivalences

$$\Gamma_{111} = \frac{1}{6} C_{111}, \quad \Gamma_{12} = \frac{1}{2} (C_{112} - C_{111}), \quad \Gamma_3 = \frac{1}{2} (C_{111} - 3C_{112} + 2C_{123}). \quad (\text{A.24})$$

The connection with other notations for third order moduli can be inferred from the table in Green's review [28] which compares many different systems of notation. We note for future reference the following formulae

$$W^{ijklmn} n_i n_j n_k n_l = 6\Gamma_{111} \delta^{mn} + 2\Gamma_{12} (\delta^{mn} - n^m n^n), \quad (\text{A.25a})$$

$$W^{ijklmn} e_i n_j e_k n_l = -\frac{1}{2} \Gamma_{12} \delta^{mn} - \frac{1}{2} \Gamma_3 (\delta^{mn} - n^m n^n - e^m e^n). \quad (\text{A.25b})$$

### 5.3.2 Wave speed dependence on the applied strain

For each deformed configuration there are three orthogonal directions of principal strain, and the small amplitude waves which propagate in these directions are either longitudinal or transverse. We will now derive approximative formulae for the incremental changes of the velocities of these principal waves to leading order in  $\varepsilon$ .

First, the variation in the velocity of the transverse wave with the polarization vector  $e_C$ ; follows from Eqs. (A.15) and (A.20) as

$$\begin{aligned}\delta c_{tC}^2 &= (\rho - \kappa \rho_f)^{-1} \delta W^{ijkl} e_{Ci} e_{Ck} n_j n_l \\ &= (\rho - \kappa \rho_f)^{-1} (W^{ijklmn} e_{Ci} e_{Ck} n_j n_l \varepsilon_{mn} + W_{\zeta}^{ijkl} e_{Ci} e_{Ck} n_j n_l \varepsilon \Delta).\end{aligned}\quad (\text{A.26})$$

Note that the densities  $\rho$  and  $\rho_f$  are unchanged because we are using the reference or Lagrangian description, as opposed to the current or Eulerian description. Conservation of mass requires that these densities are constant. The structure constant  $\kappa$  is also assumed to remain the same under the deformation. Combining Eqs. (A.20), (A.25) and (A.26) we arrive at the formula for the elasto-acoustic effect for transverse waves,

$$\delta c_{tC}^2 = \frac{-\varepsilon}{2(\rho - \kappa \rho_f)} ((\alpha_1 + \alpha_2 + \alpha_N) \Gamma_{12} + (\alpha_1 + \alpha_2 - \alpha_C) \Gamma_3 + \Gamma_{2\zeta} \Delta). \quad (\text{A.27})$$

In particular, the split in the velocities of two transverse waves with the same propagation direction but different polarizations is, from Eq. (A.27),

$$\frac{\delta c_{tA}^2 - \delta c_{tB}^2}{\alpha_A - \alpha_B} = \frac{\varepsilon \Gamma_3}{2(\rho - \kappa \rho_f)}. \quad (\text{A.28})$$

In order to establish analogous formulae for longitudinal principal waves we first write the system of Eq. (A.5) as

$$\mathbf{AX} = c^2 \mathbf{CX}, \quad (\text{A.29})$$

where  $\mathbf{X} = (h, H)^T$ , and the symmetric matrices  $\mathbf{A}$ ,  $\mathbf{C}$  are

$$\mathbf{A} = \begin{bmatrix} W^{ijkl} n_i n_j n_k n_l & -W_{\zeta}^{il} n_i n_l \\ -W_{\zeta}^{il} n_i n_l & W_{\zeta\zeta} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \rho & \rho_f \\ \rho_f & \kappa^{-1} \rho_f \end{bmatrix}. \quad (\text{A.30})$$

The inertial matrix  $\mathbf{C}$  is again unchanged by the initial deformation, and the new wave speeds depend upon the variation in  $\mathbf{A}$ . By making use of Eqs. (A.18) through (A.23) we can approximate this to within first order in  $\varepsilon$ ,

$$\mathbf{A} = \mathbf{A}^o + \varepsilon \mathbf{A}^1, \quad (\text{A.31})$$

where

$$\mathbf{A}^o = \begin{bmatrix} \lambda_c + 2\mu & \alpha M \\ \alpha M & M \end{bmatrix}, \quad \mathbf{A}^1 = \alpha_N \begin{bmatrix} 6\Gamma_{111} & -2\Gamma_{11\zeta} \\ -2\Gamma_{11\zeta} & 2\Gamma_{1\zeta\zeta} \end{bmatrix} + (\alpha_1 + \alpha_2) \begin{bmatrix} 6\Gamma_{111} + 2\Gamma_{12} & -2\Gamma_{11\zeta} - \Gamma_{2\zeta} \\ -2\Gamma_{11\zeta} - \Gamma_{2\zeta} & 2\Gamma_{1\zeta\zeta} \end{bmatrix} + \Delta \begin{bmatrix} 2\Gamma_{11\zeta} & -2\Gamma_{1\zeta\zeta} \\ -2\Gamma_{1\zeta\zeta} & 6\Gamma_{\zeta\zeta\zeta} \end{bmatrix}. \quad (A.32)$$

The eigenvalue problem for the matrix  $\mathbf{C}^{-1}\mathbf{A}^o$  was first studied by Biot in 1956 (see Paper 8 of ref. [7]) who established the existence of two different longitudinal modes: the “fast” and “slow” waves with velocities  $c_f$  and  $c_s$ , respectively. The eigenvectors  $\mathbf{X}_f$  and  $\mathbf{X}_s$ , associated with these velocities can be calculated by making use of the following equation implied by Eq. (A.13),

$$\frac{\rho c^2 - (\lambda_c + 2\mu)}{\kappa^{-1} \rho_f c^2 - M} = \frac{H^2}{h^2}. \quad (A.33)$$

Corrections to the Biot fast and slow velocities due to the incremental deformations can now be calculated using standard perturbation theory, leading to the formula:

$$\delta c^2 = \epsilon \frac{\mathbf{A}_{ij}^1 \mathbf{X}^{oi} \mathbf{X}^{oj}}{\mathbf{C}_{ij} \mathbf{X}^{oi} \mathbf{X}^{oj}}. \quad (A.34)$$

Equations (A.27), (A.32), and (A.34) show that by measurement of elasto-acoustic effects we can, in principle, find the third order elastic moduli by solving a system of linear algebraic equations. In the next section we will examine some possible experimental configurations with this purpose in mind.

## 5.4 The elasto-acoustic effect in experiments

### 5.4.1 Experimental nomenclature and moduli

The application of the previous ideas to a poroelastic sample requires the ability to impose various types of stress states if all seven third order moduli are to be determined, *i.e.*,  $\{\Gamma_{111}, \Gamma_{12}, \Gamma_3, \Gamma_{1\zeta\zeta}, \Gamma_{11\zeta}, \Gamma_{2\zeta}\}$ . We first discuss the types of measurements employed in determining the four second order moduli  $\{\Gamma_{11}, \Gamma_2, \Gamma_{1\zeta}, \Gamma_{\zeta\zeta}\}$  or  $\{\lambda_c, \mu, \alpha, M\}$ . The second order moduli are related to one another *via* Eq. (A.22), and yield the classic Biot theory for linear poroelasticity, for which [38]

$$\tau_{ij} = \lambda_c u_{,k}^k \delta_{ij} + \mu(u_{i,j} + u_{j,i}) - \alpha M \zeta \delta_{ij}, \quad (A.35a)$$

$$p = -\alpha M u_{,k}^k + M \zeta. \quad (A.35b)$$

Note that  $p$  is the pore pressure, and  $\tau_{ij}$  is the total or confining stress. The effective stress,  $\tau_{ij} + \alpha p \delta_{ij}$ , is also commonly used. For example, if a hydrostatic confining stress  $\tau_{ij} = -p_c \delta_{ij}$

is applied, then the effective pressure is  $p_e = p_c - \alpha p$ . The main point is that there are two independent stress variables, or pressures if the system is hydrostatically loaded. Table 1 lists and defines three distinct bulk moduli (inverse of compressibility),  $K_c$ ,  $K$ , and  $K_M$ , associated with hydrostatic deformation of undrained, jacketed, and unjacketed samples, respectively. We also define the bulk modulus for the fluid alone,  $K_f$ , and the "coefficient of fluid content"  $\gamma$  [23, 38],

$$\frac{1}{K_f} = \frac{-1}{V_f} \frac{\partial V_f}{\partial p}, \quad \gamma = \phi \left( \frac{1}{K_f} - \frac{1}{K_M} \right), \quad (A.36)$$

in terms of which the Gassmann relation [38] is,

$$\gamma = \frac{1}{M} - \frac{\alpha}{K_M}. \quad (A.37)$$

Different types of experiments have been used to determine the second order elasticities of fluid-saturated poroelastic media. We shall discuss them briefly in order to avoid possible confusion in terminology. The terms "open" and "closed", "jacketed" and "unjacketed", and "drained" and "undrained" are all relevant to the problem, but have different connotations. We will next describe the four states of deformation illustrated in Figure 5.1, and then consider their mathematical description, and their implications for elasto-acoustic measurements.

(a) The "closed-pore" system corresponds to constancy of the fluid content or mass, and in the geometrically linear approximation this implies that  $\zeta = 0$ . The closed-pore experiment is the simplest for theoretical analysis but is rather difficult to realize in practice, at least for a high level of wetting. Conceptually, it requires that the sample is covered by an impervious closed deformable "jacket", see Figure 5.1a. (b) Alternatively, the "open" porous system, in the terminology of Biot and Willis [23], corresponds to the case of constant fluid pressure, or  $p = 0$ . However, it is more consistent to refer to this as a "drained" test and to treat it as a special case of a more general "open" configuration with  $p = \text{const}$ . Such an experiment is sketched in Figure 5.1b and is regarded as the "jacketed" test. The amount of absorbed liquid is not fixed in this experiment since it can leave the sample through the tube connected with the air. The "unjacketed" tests are sketched in Figure 5.1c and Figure 5.1d. Although the system as a whole is again contained within an impervious jacket, the porous solid sample can now exchange fluid with the surrounding free fluid in the reservoir. (c) A "conventional" "unjacketed" test is shown in Figure 5.1c. In this case the hydrostatic stress on the solid is caused by pressurizing the surrounding fluid, and therefore both the skeleton and the fluid are hydrostatically stressed with the same pressure,  $\tau_{ij} = -p_c \delta_{ij}$  and  $p = p_c$ . This type of conventional "unjacketed" test does not permit tri-axial deformation of the sample, and is thus too restrictive for the acoustoelastic effect. (d) The experiment sketched in Figure 5.1d might be helpful in avoiding this drawback, although it is presumably rather difficult to realize practically. In what follows we call it a "tri-axial unjacketed" test, and it is characterized by a constant pressure  $p$  in the pore fluid, and a tri-axial state of stress in the skeleton, implying three independent elements for  $\tau_{ij}$ .

The four states outlined above are as follows. First, for the "closed-pore" jacketed test we

have (see Table 1 for definitions)

$$\delta\zeta = 0 \Leftrightarrow \frac{p_c}{K_c} = \frac{p_e}{K}. \quad (\text{A.38})$$

In the “open-pore” jacketed test, on the other hand,

$$p = 0 \Leftrightarrow \delta\zeta = -\alpha \frac{p_c}{K}. \quad (\text{A.39})$$

In the “conventional” unjacketed test the skeleton is in a state of hydrostatic stress, experiencing the same pressure as the fluid, *i.e.*,  $p_c = p$ . In the case at hand the skeleton deformation is purely dilatational, and Eqs. (A.35) then imply

$$\delta u_{i,j} = -\frac{p}{3K_M} \delta_{ij}, \quad \delta\zeta = \gamma p, \quad (\text{A.40})$$

where  $\gamma$  is defined in Eq. (A.36). Finally, in the tri-axial “unjacketed” test both the fluid pressure and the deformation of the skeleton are simultaneously kept under control. In this case we can express the fluid dilatation parameter in terms of these as

$$\delta\zeta = \alpha \delta u_{,k}^k + \frac{p}{M}. \quad (\text{A.41})$$

#### 5.4.2 Wave speeds for specific experimental configurations

The above formulae together imply specific relations for the change in the velocities of the three wave types: transverse, and fast and slow longitudinal waves. For the first three cases below, the loading is hydrostatic, defined by the confining pressure  $p_c$ . The change in the transverse speed is therefore independent of its polarization direction.

(a) The “closed-pore jacketed” tests: for the transverse wave,

$$\frac{dc_t^2}{dp_c} = \frac{\Gamma_{12} + \frac{1}{3}\Gamma_3}{2K_c(\rho - \kappa\rho_f)}, \quad (\text{A.42})$$

and for the fast and slow waves,

$$\frac{dc^2}{dp_c} = \frac{\mathbf{B}_{ij} \mathbf{X}^{oi} \mathbf{X}^{oj}}{\mathbf{C}_{ij} \mathbf{X}^{oi} \mathbf{X}^{oj}}, \quad (\text{A.43})$$

where

$$\mathbf{B} = \frac{1}{K_c} \begin{bmatrix} -6\Gamma_{111} - \frac{4}{3}\Gamma_{12} & 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} \\ 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} & -2\Gamma_{1\zeta\zeta} \end{bmatrix}. \quad (\text{A.44})$$

(b) The “open-pore jacketed” tests:

$$\frac{dc_t^2}{dp_c} = \frac{\Gamma_{12} + \frac{1}{3}\Gamma_3 + \alpha\Gamma_{2\zeta}}{2K(\rho - \kappa\rho_f)}. \quad (\text{A.45})$$

The variation in the fast and slow wave speeds is given by Eq. (A.43), where now

$$\mathbf{B} = \frac{1}{K} \begin{bmatrix} -6\Gamma_{111} - \frac{4}{3}\Gamma_{12} - 2\alpha\Gamma_{11\zeta} & 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} + 2\alpha\Gamma_{1\zeta\zeta} \\ 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} + 2\alpha\Gamma_{1\zeta\zeta} & -2\Gamma_{1\zeta\zeta} - 6\alpha\Gamma_{\zeta\zeta\zeta} \end{bmatrix}. \quad (\text{A.46})$$

(c) The conventional “unjacketed” tests:

$$\frac{dc_t^2}{dp_c} = \frac{\Gamma_{12} + \frac{1}{3}\Gamma_3 - \gamma K_M \Gamma_{2\zeta}}{2K_M(\rho - \kappa\rho_f)}. \quad (\text{A.47})$$

The fast and slow speed changes are given by Eq. (A.43) with

$$\mathbf{B} = \frac{1}{K_M} \begin{bmatrix} -6\Gamma_{111} - \frac{4}{3}\Gamma_{12} - 2\gamma K_M \Gamma_{11\zeta} & 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} + 2\gamma K_M \Gamma_{1\zeta\zeta} \\ 2\Gamma_{11\zeta} + \frac{2}{3}\Gamma_{2\zeta} + 2\gamma K_M \Gamma_{1\zeta\zeta} & -2\Gamma_{1\zeta\zeta} - 6\gamma K_M \Gamma_{\zeta\zeta\zeta} \end{bmatrix}. \quad (\text{A.48})$$

(d) The “unjacketed” tri-axial tests: The three principal strains  $\alpha_N$ ,  $\alpha_1$ ,  $\alpha_2$ , and the pore pressure  $p \equiv \epsilon P$  are now independent parameters, and, according to Eq. (A.41),

$$\Delta = \alpha(\alpha_N + \alpha_1 + \alpha_2) + \frac{P}{M}. \quad (\text{A.49})$$

The change in the three wave speeds are then given by the general formulae in Eqs. (A.27), (A.32) and (A.34).

## 5.5 Discussion and conclusions

The acoustoelastic effect is the simplest nonlinear elastic effect which can be used to determine nonlinear elastic moduli. To this end one should measure changes of velocities of acoustic waves induced by prestrain. Although it is of higher order as compared to linear effects, acoustoelasticity can be reliably detected in sedimentary materials since the nonlinear elastic moduli are of several orders higher than their linear moduli [21]. The experiments can be performed either on core samples or *in situ*; for instance, by experimental measurement of strain induced changes in seismic waves velocities due to the solid Earth tides [18]. In fact, it is found that the *in situ* nonlinearity is as large or larger than observed in the laboratory, and may be due to the large scale heterogeneity in the Earth [18]. Such measurements can provide valuable information about the Earth’s interior, just as measurements of the velocities themselves allow one to determine linear moduli.

In the present Chapter we have established relations for the strain induced changes of acoustic wave speeds in fluid-filled poroelastic media. We have concentrated on the longitudinal and transverse waves in isotropic poroelastic substances, which become slightly anisotropic under triaxial prestrain. These changes depend not only on the prestrain of the solid skeleton but also on the drainage regime of the fluid component. Thus, the acoustoelastic effect is different for

drained, undrained, jacketed, and unjacketed tests, and explicit formulae for each configuration are described.

The 7 nonlinear elastic moduli characterizing an isotropic poroelastic substance can be found, in principle, by a sequence of acoustoelastic tests and using the formulae presented here. Thus, the nonlinear elastic modulus  $\Gamma_3$  may be determined directly using the rather simple formula of Eq. (A.28) for the split in the velocities of two transverse waves having the same direction of propagation but different polarizations. Then, using the formulae in Eqs. (A.42) and (A.45) one can determine the nonlinear moduli  $\Gamma_{12}$  and  $\Gamma_{2\zeta}$  by making shear wave measurements under uniform (hydrostatic) prestrain with “closed-pore jacketed” and “open-pore unjacketed” conditions. The same information can be extracted using shear wave data from a conventional “unjacketed” test and Eq. (A.47). Compatibility of the numerical results for the moduli determined through different experiments can be used as the indication of acceptability of the model of an isotropic poroelastic substance. The determination of the remaining 4 nonlinear moduli can be based on measurements of the “fast” and “slow” longitudinal waves, using Eqs. (A.34), (A.43), (A.44), (A.46) and (A.48). Again, the fact that there are more relations than moduli allows one to check the admissibility of the model of an isotropic poroelastic medium.

Table I

Compressibility - definition	In current notation	Terminology, notation and references
$\frac{1}{K_c} = \frac{-1}{V} \frac{\partial V}{\partial p_c} \Big _{\zeta}$	$\frac{1}{\lambda_c + \frac{2}{3}\mu}$	“closed” compressibility, [38] undrained compressibility, $c_u$ [39]
$\frac{1}{K} = \frac{-1}{V} \frac{\partial V}{\partial p_c} \Big _p$	$\frac{1}{\lambda + \frac{2}{3}\mu}$	open or jacketed compressibility, [38, 23] drained or matrix compressibility, $c$ [39]
$\frac{1}{K_M} = \frac{-1}{V} \frac{\partial V}{\partial p_c} \Big _{p=p_c}$	$\frac{1-\alpha}{\lambda + \frac{2}{3}\mu}$	grain or pore compressibility, [40], $c_s$ [39] unjacketed compressibility, $\delta$ [23]

Table 1. Different compressibilities of a porous medium, with definitions of  $K_c$ ,  $K$  and  $K_M$ . Here,  $p$  and  $p_c$  are the pore and confining pressure, respectively. Note that  $\lambda_c$ ,  $\mu$ ,  $\alpha$  and  $M$  are defined in Eq. (A.22), and  $\lambda = \lambda_c - \alpha^2 M$  [38]. The relations between these and other moduli are explored in greater detail by Kümpel [39], and also by Brown and Korringa [40].

## Chapter 6

# Waves and flutter in fluid conveying elastic media

A fluid flows through a periodically layered system comprising alternating elastic solid and liquid constituents. An exact analysis is performed to consider the existence and stability of small acoustic waves and disturbances. The presence of the flow introduces the possibility of flow induced flutter. Unstable waves are generally possible for  $M$  of order unity, where  $M$  is the shear wave Mach number, but appear for much lower values for asymmetric flexural type motion. In that case it is found that a critical wavenumber exists, indicating that the system is inherently unstable to long wavelength disturbances.

### 6.1 Introduction

Fluid-saturated permeable elastic solids exhibit distinct phenomena not seen in either the solid or the liquid phase. The most interesting acoustic feature is the appearance of the Biot “slow” wave, predicted by Biot (1956) and observed by Plona (1980) in water saturated sintered beads. Similar acoustic effects have since been seen in other systems, including layered periodic solid-liquid configurations (Plona et al. 1987). The periodicity of the layered structure permits a more precise theoretical analysis than for a disordered system, and very good agreement has been observed between theory (Rytov 1956, Schoenberg 1983) and experiment (Plona et al. 1987). Acoustics of such layered solid-liquid structures is of great interest in various applications relating to physics, mechanical engineering, Earth science, etc. (Brekhovskikh 1981), because the structure displays the essential features of a realistic permeable elastic medium. In all previous studies of this system it was assumed that both the solid and fluid constituents are at rest in the “ground” equilibrium configuration. On the other hand, it is well-known that sufficiently fast fluid motion through a deformable solid structure can destabilize the system causing flutter-like phenomena (Bolotin 1963). So far flutter has been studied using engineering theories of plates and shells for the solid phase, combined with hydrodynamic theories for the liquid, which are sometimes based on the hypothesis of plane cross-sections, rather than with the help of much more reliable theories of bulk hydrodynamics and elasticity (Fung 1955, Bolotin 1963).

The purpose of this Chapter is to report on new results and physical phenomena caused by the non-small relative velocity of the fluid propagating between solid elastic layers. Some of these effects are a direct consequence of the flow, and vanish in the equilibrium, no-flow state. The study of acoustic disturbances and flutter-like phenomena in the presence of flow requires a nonlinear basis, and hence, our study begins with an exact nonlinear formulation. We then linearize the equations and interface boundary conditions in the vicinity of a stationary state in which parallel isotropic elastic layers are undeformed, while the fluid moves with the constant velocity  $V^o$ . Our objective is to explore the dependence on  $V^o$  of the dispersion equations of acoustic waves in the small vicinity of the steady, uniform flow configuration.

## 6.2 Formulation and general solution

### 6.2.1 Nonlinear equations

Let us consider a nonviscous layered media consisting of alternating solid and fluid layers. We choose the Eulerian description of continua in order to simplify exact formulation of the interface boundary conditions. We emphasize that the relative displacements of the fluid and solid constituents are not small, generally speaking. We denote by  $x^i$  the Eulerian Cartesian coordinates, and assume that the  $x^1$  coordinate is directed along the undisturbed elastic layer. The bulk equations within the solid and fluid constituents and the interface boundary conditions are then

$$\rho_s \left( \frac{\partial V_s^i}{\partial t} + V_s^j \nabla_j V_s^i \right) = \nabla_j P_s^{ji}, \quad (\text{A.1a})$$

$$V_s^i = \frac{\partial U_s^i}{\partial t} + V_s^j \nabla_j U_s^i, \quad (\text{A.1b})$$

$$\rho_f \left( \frac{\partial V_f^i}{\partial t} + V_f^j \nabla_j V_f^i \right) = -\nabla^i p_f, \quad (\text{A.1c})$$

$$\frac{\partial \rho_f}{\partial t} + \nabla_i (\rho_f V_f^i) = 0, \quad (\text{A.1d})$$

$$P_s^{ji} N_j = -p_f N^i, \quad (\text{A.1e})$$

$$V_s^j N_j = V_f^j N_j, \quad (\text{A.1f})$$

where  $\rho_s$  and  $\rho_f$  are the actual densities of the constituents,  $V_s^i$  and  $V_f^i$  are their velocities,  $U_s^i$  is the displacement of the solid,  $P_s^{ji}$  and  $p_f$  are the Cauchy stress tensor of the solid and the fluid pressure, respectively, and  $N^i$  is the unit normal of the interface (which, generally speaking, changes).

The system of equations (A.1) permits a stationary solution in which the solid layers are at rest and undeformed whereas the fluid moves with constant velocity  $V^o$  parallel to the elastic layers. Linearizing the system of equations (A.1) we arrive at the governing system for the small disturbances.

### 6.2.2 Linearization

We consider small dynamic motion superimposed on the state of uniform pressure and flow in the fluid, corresponding to fluid velocity  $V^0$ , fluid pressure  $p^0$ , and densities  $\rho_f^0$  and  $\rho_s^0$ . The solid is also assumed to be in a state of uniform static initial stress,  $P_s^{ji} = P_o^{ji}$ , either hydrostatic ( $P_o^{ji} = -p^0 \delta^{ji}$ ) or otherwise, with the initial deformation homogeneous and defined by  $U_s^i = U_o^i$ . Thus, we let

$$V_f^i = V^0 \delta^{i1} + v_f^i, \quad (\text{A.2a})$$

$$p_f = p^0 + p, \quad (\text{A.2b})$$

$$U_s^i = U_o^i + u_s^i, \quad (\text{A.2c})$$

$$P_s^{ji} = P_o^{ji} + \sigma^{ji}, \quad (\text{A.2d})$$

where  $v_f^i$  and all the remaining dynamic quantities ( $u_s^i$ ,  $V_s^i$ ,  $\sigma^{ji}$ ,  $\rho_s - \rho_s^0$ ,  $\rho_f - \rho_f^0$ , and  $p$ ) are small. We consider motion in the  $x^1 - x^3$  plane, where  $x^3$  is the coordinate in the layering direction. Then equations (A.1) implies the following linearized equations

$$\rho_s^0 \frac{\partial V_s^i}{\partial t} = \nabla_j \sigma^{ji}, \quad (\text{A.3a})$$

$$V_s^i = \frac{\partial u_s^i}{\partial t}, \quad (\text{A.3b})$$

$$\rho_f^0 \left( \frac{\partial v_f^i}{\partial t} + V^0 \frac{\partial v_f^i}{\partial x^1} \right) = -\nabla^i p, \quad (\text{A.3c})$$

$$\frac{\partial \rho_f}{\partial t} + \rho_f^0 \nabla_i V_f^i + V^0 \frac{\partial \rho_f}{\partial x^1} = 0, \quad (\text{A.3d})$$

$$\sigma^{3i} = -p \delta^{3i}, \quad (\text{A.3e})$$

$$V_s^3 + V^0 \frac{\partial u_s^3}{\partial x^1} = v_f^3. \quad (\text{A.3f})$$

The equations for the solid phase, (A.3a,b), must be supplemented by a linear stress-strain relation for the small stress  $\sigma^{ji}$ , of the form

$$\sigma_{ji} = C_{jikl} \nabla^k u_s^l. \quad (\text{A.4})$$

Therefore, eqs. (A.3a) and (A.3b) imply that  $U_s^i$  satisfies the usual equations of linear dynamic elasticity,

$$C_{jikl} \nabla^j \nabla^k u_s^l - \rho_s^0 \frac{\partial^2 u_s^i}{\partial t^2} = 0. \quad (\text{A.5})$$

The fluid equations (A.3c,d) require an additional linear equation of state,

$$\rho_f - \rho_f^0 = \frac{p}{c_f^2}, \quad (\text{A.6})$$

where  $c_f$  is the speed of sound. When combined with eqs. (A.3c) and (A.3d), this implies a convective wave equation for the small pressure,

$$\nabla_i \nabla^i p - \frac{1}{c_f^2} \left( \frac{\partial}{\partial t} + V^0 \frac{\partial}{\partial x^1} \right)^2 p = 0. \quad (\text{A.7})$$

### 6.2.3 Traveling wave solutions

Let us consider “traveling-wave”-solutions of the linearized system, i.e., the solutions proportional to  $e^{i(kx^1 - \omega t)}$ . There are two sorts of such solutions which we call symmetric and asymmetric, respectively. For the symmetric mode the shape of each layer remains symmetric with respect to its median. The dispersion equation for this mode is derived below as

$$\left( 2 - \frac{c^2}{c_t^2} \right)^2 \frac{\tanh(k\xi_t H_s)}{\tanh(k\xi_l H_s)} - 4\xi_l \xi_t + \frac{\rho_f^0}{\rho_s^0} \left( 1 - \frac{V^0}{c} \right)^2 \frac{\xi_l c^4}{\xi_f c_t^4} \frac{\tanh(k\xi_t H_s)}{\tanh(k\xi_f H_f)} = 0. \quad (\text{A.8})$$

Here we use the following notation:  $c = \omega/k$  is the velocity of the traveling wave,  $2H_s$  and  $2H_f$  are the equilibrium thicknesses of the fluid and solid layers, respectively,  $\xi_l = (1 - c^2/c_l^2)^{1/2}$ ,  $\xi_t = (1 - c^2/c_t^2)^{1/2}$ , and  $\xi_f = (1 - (c - V^0)^2/c_f^2)^{1/2}$ , where  $c_t$  and  $c_l$  are the velocities of bulk transverse and longitudinal waves, respectively, within the undeformed solid layer, and  $c_f$  is the bulk sound velocity within the undisturbed fluid. Also,  $\rho_s^0$  and  $\rho_f^0$  are the undisturbed densities.

In the asymmetric mode opposite edge points of each layer have the same vertical velocity. The dispersion equation for the asymmetric mode is the following:

$$\left( 2 - \frac{c^2}{c_t^2} \right)^2 \frac{\tanh(k\xi_l H_s)}{\tanh(k\xi_t H_s)} - 4\xi_l \xi_t + \frac{\rho_f^0}{\rho_s^0} \left( 1 - \frac{V^0}{c} \right)^2 \frac{\xi_l c^4}{\xi_f c_t^4} \frac{\tanh(k\xi_f H_f)}{\tanh(k\xi_t H_s)} = 0. \quad (\text{A.9})$$

### 6.2.4 Derivation

The dispersion relations can be derived using impedance-type concepts. Consider the layer of solid of thickness  $2H_s$ , subject to a normal stress  $\sigma^{33} = \sigma_0 e^{i(kx^1 - \omega t)}$  on either face such that the motion is either symmetric or asymmetric. The shear stress on both faces is zero. Similarly, consider the moving fluid layer subject to a pressure disturbance  $p = p_0 e^{i(kx^1 - \omega t)}$ , again either symmetric or asymmetric. The symmetry or asymmetry means we need only consider half of the unit period of the system, that is, the solid and fluid half layers in  $-H_s < x^3 < 0$  and  $0 < x^3 < H_f$ , respectively. For each half-layer problem, let  $v_0^3 e^{i(kx^1 - \omega t)}$  be the vertical velocity  $v_f^3$  or  $V_s^3$  at the surface  $x^3 = 0$ . Define the effective impedances,

$$Z_s^{(\alpha)} \equiv \frac{\sigma_0}{v_0^3}, \quad Z_f^{(\alpha)} \equiv \frac{p_0}{v_0^3}, \quad (\text{A.10})$$

where  $\alpha = \pm 1$  indicates the symmetry. Thus,  $\alpha = 1$  and  $-1$  correspond to the symmetric and asymmetric configurations, respectively.

In the absence of the flow the dispersion equations for guided waves of the system are defined by the identity  $Z_s^{(\alpha)} + Z_f^{(\alpha)} = 0$ , which follows from the continuity of traction and normal velocity at the interface. That is, from eqs. (A.3e) and (A.3f) with  $V^\circ = 0$ ,

$$\sigma^{33} = -p \quad \text{and} \quad V_s^3 = v_f^3 \quad \text{at} \quad x^3 = 0. \quad (\text{A.11})$$

However, in the presence of flow the kinematic constraint of equation (A.11)<sub>2</sub> must be replaced by (A.3f). For the particular dependence  $e^{i(kx^1 - \omega t)}$  this simplifies to

$$v_f^3 = \left(1 - \frac{V^\circ}{c}\right) V_s^3 \quad \text{at} \quad x^3 = 0. \quad (\text{A.12})$$

Thus, the dispersion relation in the presence of flow becomes

$$Z_s^{(\alpha)} + \left(1 - \frac{V^\circ}{c}\right) Z_f^{(\alpha)} = 0, \quad \alpha = \pm 1. \quad (\text{A.13})$$

A simple calculation based on equations (A.7) and (A.3c) shows that

$$Z_f^{(\pm 1)} = -i \frac{\rho_f^\circ c}{\xi_f} \left(1 - \frac{V^\circ}{c}\right) (\coth(k\xi_f H_f))^{\pm 1}. \quad (\text{A.14})$$

For simplicity, we assume the solid layer to be isotropic, in which case standard analysis gives

$$Z_s^{(\pm 1)} = i \frac{\rho_s^\circ c_t^4}{c^3 \xi_l} \left(4\xi_l \xi_t (\coth(k\xi_t H_s))^{\pm 1} - \left(2 - \frac{c^2}{c_t^2}\right)^2 (\coth(k\xi_l H_s))^{\pm 1}\right). \quad (\text{A.15})$$

## 6.3 Asymptotic limits

### 6.3.1 Short wavelength asymptotics: two half spaces

We now consider several asymptotic cases of the general results defined by equations (A.8) and (A.9). In the short wavelength limit  $|kH_s| \sim |kH_f| \gg 1$  the dispersion equations for both modes lead to the same equation:

$$\left(2 - \frac{c^2}{c_t^2}\right)^2 - 4\xi_l \xi_t + \frac{\rho_f^\circ}{\rho_s^\circ} \left(1 - \frac{V^\circ}{c}\right)^2 \frac{\xi_l}{\xi_f} \frac{c^4}{c_t^4} = 0. \quad (\text{A.16})$$

By introducing nondimensional parameters  $M = V^\circ/c_t$ ,  $s = c/c_t$ ,  $s_l = c_l/c_t$ ,  $s_f = c_f/c_t$ , and  $\delta = \rho_f^\circ/\rho_s^\circ$ , we can rewrite the last equation in the following form:

$$\left(2 - s^2\right)^2 - 4\sqrt{1 - s^2} \sqrt{1 - \frac{s^2}{s_l^2}} + \delta s^2 (s - M)^2 \frac{\sqrt{1 - \frac{s^2}{s_l^2}}}{\sqrt{1 - \frac{(s - M)^2}{s_f^2}}} = 0. \quad (\text{A.17})$$

This is identical with the Rayleigh equation for  $\rho_f^0 = 0$  (Rayleigh 1885, Achenbach 1973), and it gives the Scholte equation for  $V^0 = 0$  (Scholte 1948, 1949). By solving it numerically one can find that the roots for the interface wave speed  $c = \omega/k$  become complex and the system becomes unstable for  $V^0/c_t = O(1)$ . For example, Figure 6.1 shows the merging of two real roots as  $M$  is increased from zero. In this example, complex roots appear for  $V^0$  greater than about  $1.793 c_t$ .

### 6.3.2 Long wavelength asymptotics: symmetric mode

The long wavelength asymptotics  $|kH_s| \sim |kH_f| \ll 1$  appears to be quite different for the symmetric and asymmetric modes. For the symmetric mode we get

$$\left(2 - \frac{c^2}{c_t^2}\right)^2 - 4\xi_l^2 + \frac{\rho_f^0}{\rho_s^0} \left(1 - \frac{V^0}{c}\right)^2 \frac{\xi_l^2 c^4}{\xi_f^2 c_t^4} \frac{H_s}{H_f} = 0, \quad (\text{A.18})$$

or in non-dimensional form

$$\left(2 - s^2\right)^2 - 4 \left(1 - \frac{s^2}{s_l^2}\right) + \frac{\delta}{h} s^2 (s - M)^2 \left(\frac{1 - \frac{s^2}{s_l^2}}{1 - \frac{(s - M)^2}{s_f^2}}\right) = 0, \quad (\text{A.19})$$

where  $h = H_f/H_s$ . This can be rewritten as a fourth order polynomial equation for  $s$ ,

$$\left((s - M)^2 - s_f^2\right) \left(s^2 - s_p^2\right) \frac{\phi}{\rho_f^0 s_f^2} + (s - M)^2 \left(s^2 - s_l^2\right) \frac{(1 - \phi)}{\rho_s^0 s_l^2} = 0, \quad (\text{A.20})$$

where  $\phi = H_f/(H_f + H_s)$  is the porosity, and  $s_p = c_p/c_t$  where  $c_p$  is the speed of a longitudinal “plate” wave; thus,  $s_p^2 = 2/(1 - \nu)$ . We note that the symmetric waves for the long wavelength asymptotic limit of an isolated plate in a fluid,  $|kH_s| \ll 1, |kH_f| \rightarrow \infty$ , are contained in equation (A.20) as the limiting case  $\phi \rightarrow 1$ .

At  $M = 0$  we get a quadratic equation with respect to  $s^2$  which has two physically meaningful roots: the velocities of the so-called Biot “fast” and “slow” waves (Rytov 1956, Schoenberg 1983, 1984, Plona et al. 1987). The velocities of both waves do not depend on the direction of propagation. The non-zero relative velocity  $V^0$  of the fluid destroys the equivalence of the opposite directions. For small  $M$  the last equation allows one to find the magnitude of the velocity splitting for each type of wave. Thus, let  $s_0 > 0$  be the non-dimensional wave speed relative to  $c_t$  for  $M = 0$ , fast or slow, and let  $s_1$  be the other speed, slow or fast, then the split velocities are  $\pm s_0 + \beta M$ , where

$$\beta = \frac{\left(s_0^2 - s_p^2\right) \frac{\phi}{\rho_f^0 s_f^2} + (s_0^2 - s_l^2) \frac{(1 - \phi)}{\rho_f^0 s_l^2}}{\left(s_0^2 - s_1^2\right) \left(\frac{\phi}{\rho_f^0 s_f^2} + \frac{(1 - \phi)}{\rho_f^0 s_l^2}\right)}. \quad (\text{A.21})$$

The Biot fast and slow wave roots for  $M = 0$  simplify when both constituents are incompressible, that is  $s_l, s_f \rightarrow \infty, s_p = 2$ . The fast speed then becomes infinite but the slow roots

are

$$c \approx \pm \frac{c_p}{\sqrt{1 + \frac{\delta}{h}}} + \frac{V^\circ \delta}{\delta + h}. \quad (\text{A.22})$$

### 6.3.3 Asymmetric modes - flexural waves

In the asymptotic limit of  $|kH_s| \ll 1$  the dispersion equation (A.9) for the asymmetric mode gives

$$s^2 - \frac{2(kH_s)^2}{3(1-\nu)} + \delta h (s - M)^2 \frac{\tanh(kH_f \xi_f)}{kH_f \xi_f} = 0. \quad (\text{A.23})$$

Flexural type waves are found using the scaling  $c = c_t kH_s r$ , where  $r$  is a parameter of order unity. Substituting this into equation (A.23) yields,

$$r^2 - \frac{s_p^2}{3} + \delta h \left( r - \frac{M}{kH_s} \right)^2 \frac{\tanh(kH_f \xi_f)}{kH_f \xi_f} = 0. \quad (\text{A.24})$$

The flexural wave dispersion relation for a plate *in vacuo* follows from this with  $\delta = 0$  as  $c = \pm c_b$  where  $c_b = c_t |kH_s| \sqrt{2/3(1-\nu)}$  is the phase speed of a flexural or bending wave on a plate. The dispersion relation for a fluid-loaded plate in the absence of flow is obtained from (A.24) by setting  $M = 0$ . We have retained the parameter  $\xi_f$  in (A.24) rather than set it to unity in order to be consistent with standard analyses for fluid loaded plates, e.g. Junger and Feit (1986).

If the fluid layer is also very thin compared with the wavelength, i.e.  $|kH_f| \ll 1$ , then equation (A.24) reduces to a quadratic equation in  $r$ , yielding

$$c = \frac{V^\circ \delta h}{1 + \delta h} \pm \frac{c_b}{1 + \delta h} \sqrt{1 + \delta h(1 - \epsilon^2)}, \quad (\text{A.25})$$

where

$$\epsilon = \frac{M\sqrt{3}}{|kH_s|s_p} = \frac{V^\circ}{c_b}. \quad (\text{A.26})$$

When  $M = 0$ , the roots yield  $c = \pm c_b/\sqrt{1 + \delta h}$ , which correspond to flexural waves on an isolated plate which has the bending stiffness of a single elastic layer, and the mass of a single period of the solid-liquid system. That is, the effect of the fluid is just an added mass as it moves in phase with the flexural motion. For small values of  $M$ , or equivalently  $\epsilon$ , the two roots are regular perturbations of the flexural wave roots. For large  $\epsilon$ , on the other hand, we have the possibility of two complex roots when the discriminant of equation (A.25) goes to zero. The existence of complex-conjugate roots indicates that the flow causes a flutter-like instability. This occurs for  $\epsilon > \epsilon_c$  where the critical value is

$$\epsilon_c = \sqrt{1 + (\delta h)^{-1}}. \quad (\text{A.27})$$

In summary, anti-symmetric disturbances of the layered system are unstable for very long ( $\epsilon \gg 1$  or  $|kH_s| \ll M$ ) wavelength, but short wavelengths ( $\epsilon \ll 1$ ) are stable. There is a critical wavelength above which all disturbances are unstable, and it is defined by  $\epsilon = \epsilon_c$ , or  $k = k_c$ , where

$$k_c = \frac{V^\circ}{c_t H_s} \sqrt{\frac{2}{3(1-\nu)}} \left(1 + \frac{\rho_s^\circ H_s}{\rho_f^\circ H_f}\right)^{-\frac{1}{2}}. \quad (\text{A.28})$$

Finally, in order to analyze flutter of an isolated elastic layer we let  $|kH_f| \rightarrow \infty$  in the dispersion equation (A.23) for the asymmetric mode. Using the same variables as before, we consequently obtain the following equation:

$$s^2 - \frac{2(kH_s)^2}{3(1-\nu)} + \frac{\delta(s-M)^2}{|kH_s|\xi_f} = 0. \quad (\text{A.29})$$

When the fluid is incompressible this equation becomes a quadratic with roots

$$c = \frac{V^\circ \delta}{\delta + |kH_s|} \pm \frac{c_b}{\delta + |kH_s|} \sqrt{1 + \frac{\delta(1-\epsilon^2)}{|kH_s|}}, \quad \text{incompressible fluid.} \quad (\text{A.30})$$

The possibility of complex conjugate roots again shows that flutter instability occurs for long wavelength perturbation of the system. That is, the system is stable (unstable) for  $|k| > k_c$  ( $|k| < k_c$ ) where the finite wavenumber which defines the onset of the flutter regime is

$$k_c = \frac{\delta}{H_s} \lambda, \quad (\text{A.31})$$

and  $\lambda$  is the unique positive root of

$$\lambda^3 + \lambda^2 = \frac{3V^\circ 2}{c_p^2 \delta^2}. \quad (\text{A.32})$$

The case of an isolated elastic plate in a mean flow has received extensive coverage in the literature using thin plate equations (Kirchhoff theory). For example, Crighton and Ossewell (1991) discussed the response of a thin plate in an incompressible flow to a line drive. They also provide a useful review of the literature on the stability of flexible surfaces and compliant coatings in the presence of mean flow.

## 6.4 Summary

The flow of compressible fluid through a layered medium provides a very rich system for studying the phenomenon of acoustics in fluid conveying structures. Starting from the exact nonlinear equations of motion we have derived the equations for small dynamic disturbances superimposed upon a steady flow configuration. The possible wave types for the periodically layered medium may be distinguished by their parity, symmetric and asymmetric, each of which displays quite distinct stability characteristics.

The general dispersion relation for symmetric modes is given by equation (A.8), with short and long wavelength limits in equations (A.16) and (A.18), respectively. It is found that instability is possible only for flow speed on the order of the bulk wave speeds, i.e., for  $M = O(1)$ .

Regarding asymmetric modes, the general dispersion relation of equation (A.9) has the same short wavelength limit as that for symmetric modes, equation (A.16). Thus, disturbances of short wavelength become unstable only for  $M = O(1)$ . However, disturbances of very long wavelength are potentially unstable in the presence of flow for both the periodic system, as indicated by equation (A.25), and for an isolated plate, from equation (A.30). The associated long wavelength wave types are analogous to flexural waves on plates *in vacuo*. For a given flow speed, or  $M$ , there is a critical wavenumber,  $k_c$ , such that quasi-flexural waves are stable for  $k > k_c$ , but instability is possible for all  $k < k_c$ .

## Chapter 7

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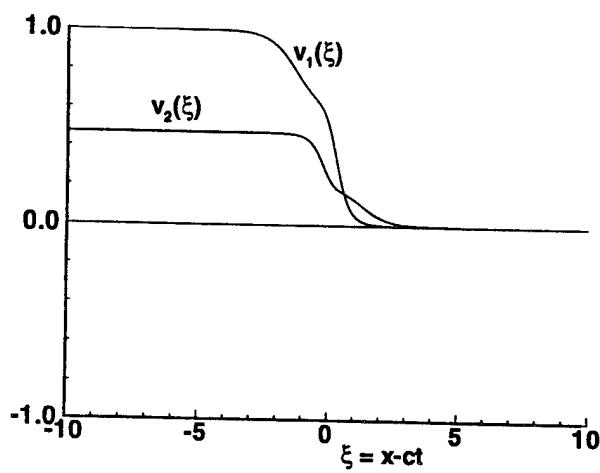
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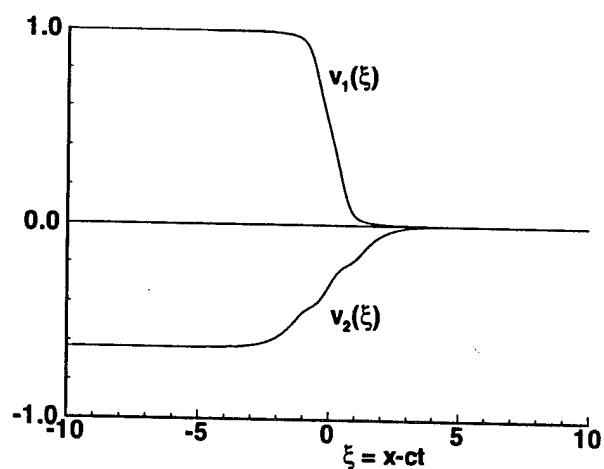
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## Chapter 8

### Figures



a fast shock front in a two liquid medium



a slow front

Figure 3.1

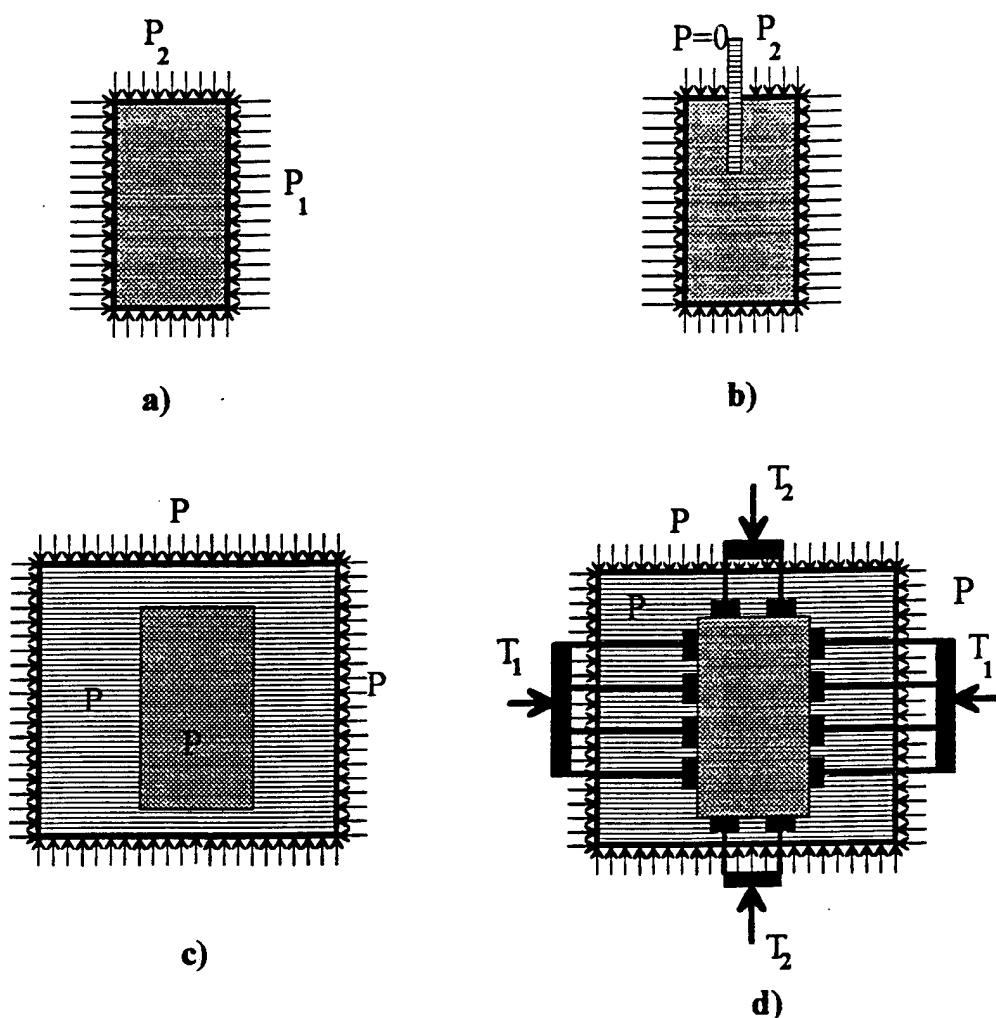


Figure 5.1 Different schemes of loading: (a) closed-pore jacketed test, (b) open-pore jacketed test, (c) conventional unjacketed test, and (d) tri-axial unjacketed test.

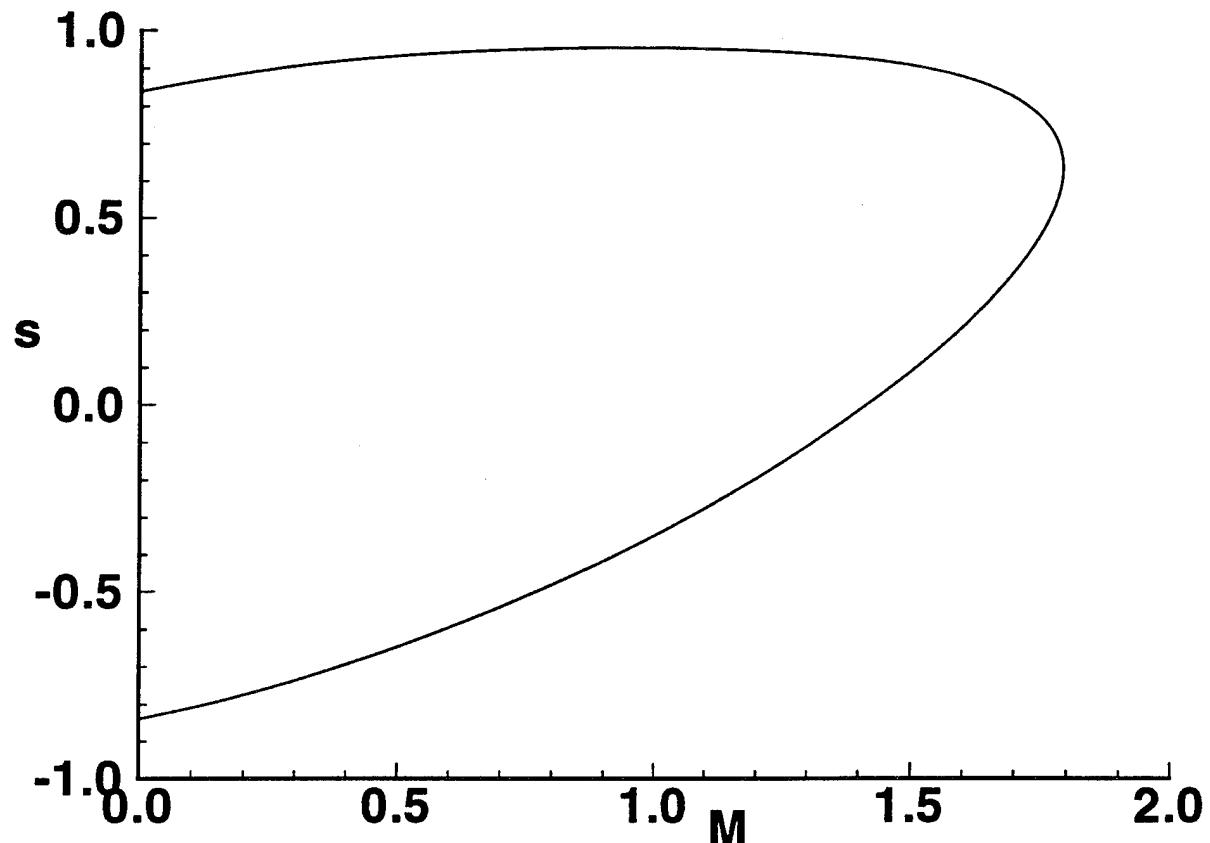


Figure 6.1 The real roots for flow over a solid half space, from equation (A.17). For simplicity, we have taken the fluid and solid as incompressible ( $s_l, s_f \rightarrow \infty$ ), and  $\delta = 1$ . The emergence of complex roots occurs at  $M \approx 1.793$  for this case.

## Chapter 9

### Attachment

A. N. Norris and M. A. Grinfeld, "Nonlinear poroelasticity for a layered medium", *J. Acoust. Soc. Am.* **98**, 1138-1146, 1995.

# Nonlinear poroelasticity for a layered medium

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The equations of motion and the nonlinear constitutive theory of fluid-filled poroelastic media are derived from the fundamental equations of elasticity and fluid mechanics for the constituents. A two-scale spatial expansion and the method of homogenization are employed. Explicit equations are obtained for the special case of a medium consisting of alternating solid and fluid layers. The linearized theory is examined in depth for the particular case of isotropic solid layers. The governing effective poroelastic medium is transversely isotropic, and wave solutions are discussed and compared with previous studies. © 1995 Acoustical Society of America.

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## INTRODUCTION

Biot's theory<sup>1,2</sup> for linear dynamics of fluid-filled porous media is by now well established, with several experimental investigations that corroborate it very well. For example, the data of Plona *et al.*<sup>3</sup> display both the fast and slow wave in a layered, alternating fluid/solid system. It has recently been emphasized that sandstone and other porous granular media<sup>4,5</sup> display strongly nonlinear behavior, as compared with, for example, water or metals. The nonlinearity parameter  $B/A$  for water, which is the ratio of a third-order elastic modulus to the second-order bulk modulus, is approximately 5. The analogous quantity for sandstone can be on the order of  $10^4$ .<sup>4</sup> What can one expect for the nonlinear behavior of a strongly nonlinear porous sandstone saturated with water? The nonlinear acoustical properties of the constituents are disparate, and it is not clear how this will affect the nonlinear acoustics of the fast and slow waves of the poroelastic medium. This paper is a first step toward an understanding of the nonlinear mechanics and dynamics of porous media, particularly the interplay between the nonlinearities of the constituents. We focus on the problem of deriving the governing equations, the strain energy functions, and describe some linear wave solutions for a layered medium.

Our objective is the governing nonlinear equations for a poroelastic medium—the nonlinear generalization of the Biot model.<sup>2</sup> We employ the “two-scale” technique of homogenization for heterogeneous media with disparate length scales, which has been used to obtain the linear Biot theory.<sup>6–10</sup> Homogenization leads directly to the macroscopic equations, and defines the microproblems which uniquely determine the coefficients in them. Burridge and Kostek<sup>11</sup> recently derived the nonlinear poroelasticity equations for a system with a granular solid skeleton whose elastic response is governed primarily by deformation at grain contacts. This is assumed to dominate the nonlinear effects, to the extent that the fluid can be considered as linearly elastic. Unlike Burridge and Kostek,<sup>11</sup> we do not assume that the solid nonlinearity overwhelms the fluid's, but keep both on an equal

footing with the purpose of comparing their interaction.

Three sources of nonlinearity are traditionally distinguished in continuum mechanics. First, there is the physical nonlinearity, that is, the nonlinearity associated with the dependence of the stress tensor on the tensor of finite deformations. The second one is the nonlinearity of the universal equations of mass, momentum, etc., that is, those equations which have validity for all specific models of the continuous medium. The third one is geometrical nonlinearity, for example, the nonlinear relationship between displacements and the tensor of finite deformation. This classification of nonlinearities is not absolute, and some changes occur when one switches from, for instance, the Eulerian to the Lagrangian description (e.g., the universal momentum equation appears to be nonlinear in the Eulerian description and linear in the Lagrangian description). In order to keep this paper as simple as possible, we choose the Lagrangian description and concentrate on the physical nonlinearity, which seems to be the most significant for water saturated sandstones. The assumption of a periodic microstructure is another significant physical assumption made for the purpose of simplifying the effects of nonlinearity.

We consider in detail a layered fluid/solid medium. The general form of the nonlinear equations is derived, but we consider only linear wave solutions in this paper. Nonlinear waves and the interplay between the solid and fluid nonlinearity will be explored in a separate paper. The stratified medium is perhaps the simplest realization of a porous medium, and has been examined in several studies, both theoretical<sup>12–17</sup> and experimental.<sup>3</sup> The first study by Rytov<sup>12</sup> considered wave motion parallel to the layers, and showed the existence of two waves: the fast and the slow. Bedford<sup>15</sup> later showed that the slow wave is the long-wavelength manifestation of the second mode of the system. This is a rich system for wave motion, particularly when the waves are allowed to propagate in oblique directions. The definitive theoretical and experimental studies of Schoenberg and co-workers<sup>3,13,14,16</sup> describe the wave motion and slowness surfaces for fast and slow waves. We give for the first time

TABLE I. Material parameters, from Ref. 3.

Material	Compressional speed (m/s)	Shear speed (m/s)	Density (kg/m <sup>3</sup> )
Aluminum	6450	3150	2700
Plexiglass	2700	1380	1200
Water	1490	...	1000

the complete set of Biot equations for the stratified fluid/solid system, which includes fluid viscosity and elastic anisotropy.

The paper proceeds as follows. The nonlinear equations of motion are outlined and the scaling procedure is defined in Sec. I. The main difficulty in any averaging theory of heterogeneous media is to calculate the “effective” parameters (see Table I), which require solving simpler problems on typical unit cells. Explicit solutions can be obtained for the specific case of the layered medium, and are addressed in Sec. II. All moduli and parameters can be found for this simple geometry; in particular, the time-dependent viscodynamic operator is explicit. The effective nonlinear equations of motion are summarized in Sec. III. Section IV focuses on the linear limit, and comparisons are made with Schoenberg’s model<sup>13,14</sup> for a layered medium of fluid and isotropic solid in Sec. V.

## I. GOVERNING EQUATIONS AND SCALINGS

### A. The primitive field equations

The volume  $V$  comprises fluid and solid regions,  $V_f$  and  $V_s$ , with boundary  $\partial V$  between them. The displacement fields are  $u_i$  and  $U_i$  in the solid and fluid, and the stress fields are  $\sigma_{ji}$  and  $\Sigma_{ji}$ , respectively. All equations are defined in terms of the reference or Lagrangian coordinates. The governing equations of motion and constitutive relations are

$$\rho_s \partial_t^2 u_i = D_j \sigma_{ji}, \quad (1a)$$

$$\sigma_{ji} = \sigma_{ji}(F_{mn}), \quad (1b)$$

in  $V_s$ ,

$$\rho_f \partial_t^2 U_i = D_j \Sigma_{ji}, \quad (2a)$$

$$\Sigma_{ji} = -p \delta_{ji} + 2 \bar{\eta} \partial_t \{D_j U_i\}, \quad (2b)$$

$$p = \mathcal{A}(D_i U_i), \quad (2c)$$

in  $V_f$ , and the continuity conditions on  $\partial V$  are

$$U_i = u_i, \quad (3a)$$

$$n_j \Sigma_{ji} = n_j \sigma_{ji}. \quad (3b)$$

Here,  $F_{ij} = D_j u_i$  is the deformation gradient tensor in the solid,  $\{a_{ij}\} = (a_{ij} + a_{ji})/2 - \delta_{ij} a_{kk}/3$  is the symmetric deviatoric part of a second-order tensor, and  $n_j$  is the unit normal on  $\partial V$  directed into the fluid region. The summation of repeated Latin suffices over 1, 2, and 3 is understood.

Both the solid and fluid constitutive relations are nonlinear. We ignore, however, the “geometrical” nonlinearity in the fluid equation (2b) for the sake of simplicity (conceptually, many of our conclusions remain valid without this assumption). Also, the equation of state for the pressure, Eq.

(2c), is normally expressed in terms of the current (Eulerian) density, but we take a slightly different approach, and assume that it depends upon the dilatation in the reference description. This is simpler, and leads to the same nonlinear effects that are normally present in homogeneous fluids up to second order. Also, we assume that the fluid equation of state possesses an inverse,

$$D_i U_i = \mathcal{Q}(p). \quad (4)$$

The material nonlinearity of the phases is then reflected in the nonlinear behavior of the functions  $\mathcal{P}$ ,  $\mathcal{Q}$ , and  $\sigma_{ji}(F_{mn})$ . The stress functions  $\sigma_{ji}$  and the pressure are normally derived from strain energy potentials, according to

$$\sigma_{ji} = \frac{\partial E_s}{\partial (D_j u_i)}, \quad -p = \frac{\partial E_f}{\partial (D_i U_i)}, \quad (5)$$

where  $E_s(D_j u_i)$  and  $E_f(D_i U_i)$  are the elastic strain energy functions for the solid and fluid, respectively. The homogenization theory outlined below is not dependent upon the existence of the strain energies, and the final equations do not involve them. However, we will see that the effective poroelastic medium also possesses a strain energy function when  $E_s$  and  $E_f$  exist.

### B. Asymptotic scaling

In order to simplify the system of equations (1)–(3) we employ the method of homogenization. We assume the medium is characterized by two distinct spatial lengths,  $h \ll H$ , associated with the micro- and macroscales, respectively. The number,  $\epsilon \equiv h/H$ , is a small parameter,  $\epsilon \ll 1$ . Furthermore, we assume that the viscosity scales as

$$\bar{\eta} = \epsilon^2 \eta. \quad (6)$$

References 7–9 and 11 provide a clear motivation for this choice. The results generated by the homogenization technique are very sensitive to the presence or absence of the multiplier  $\epsilon^2$  in the viscosity. Without it the “macro” equations turn out to be of viscoelastic type, whereas the presence of the term  $\epsilon^2$  in Eq. (6) leads to Biot-type equations, as we will demonstrate. We refer the reader to the cited papers for further details of the homogenization procedure. [See also the comment after Eq. (32).]

We assume a two-scale expansion<sup>18</sup> in terms of the slow or macroscopic spatial variable  $x$ , and the fast variable  $y = x/\epsilon$ , such that the spatial differential operator is

$$D_j = \partial_{x_j} + \epsilon^{-1} \partial_{y_j}. \quad (7)$$

The fast variable  $y$  reflects the small-scale structure in the problem through the perturbation parameter  $\epsilon$ . Our goal is to eliminate the explicit dependence on  $y$ , leaving us with equations in  $x$ . The governing equations (1)–(3), combined with Eq. (6), become

$$\epsilon \rho_s \partial_t^2 u_i = (\epsilon \partial_{x_j} + \partial_{y_j}) \sigma_{ji}, \quad (8a)$$

$$\sigma_{ji} = \sigma_{ji}[(\partial_{x_m} + \epsilon^{-1} \partial_{y_m}) u_n] \quad (8b)$$

in the solid phase, and

$$\epsilon \rho_f \partial_t^2 U_i = (\epsilon \partial_{x_j} + \partial_{y_j}) \Sigma_{ji}, \quad (9a)$$

$$\Sigma_{ji} = -p \delta_{ji} + \epsilon 2 \eta \partial_t \{ (\epsilon \partial_{x_j} + \partial_{y_j}) U_i \}, \quad (9b)$$

$$\mathcal{Q}(p) = (\partial_{x_i} + \epsilon^{-1} \partial_{y_i}) U_i \quad (9c)$$

in the fluid. We consider the *ansatz*

$$u_i(\mathbf{x}, t, \epsilon) = u_i^0(\mathbf{x}, t) + \sum_{n=1}^{\infty} \epsilon^n u_i^n(\mathbf{x}, \mathbf{y}, t), \quad (9b)$$

$$p(\mathbf{x}, t, \epsilon) = p^0(\mathbf{x}, t) + \sum_{n=1}^{\infty} \epsilon^n p^n(\mathbf{x}, \mathbf{y}, t), \quad (9c)$$

$$U_i(\mathbf{x}, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n U_i^n(\mathbf{x}, \mathbf{y}, t), \quad (10)$$

$$\sigma_{ji}(\mathbf{x}, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \sigma_{ji}^n(\mathbf{x}, \mathbf{y}, t), \quad (11)$$

$$\Sigma_{ji}(\mathbf{x}, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \Sigma_{ji}^n(\mathbf{x}, \mathbf{y}, t). \quad (12)$$

Substituting these expansions into the governing system and comparing like powers of  $\epsilon$  yields a sequence of asymptotic equations. The leading-order equations, of order unity, are

$$\partial_{y_j} \sigma_{ji}^0 = 0, \quad (11a)$$

$$\sigma_{ji}^0 = \sigma_{ji}(\partial_{x_m} u_n^0 + \partial_{y_m} u_n^1), \quad (11b)$$

for  $\mathbf{x}$  and  $\mathbf{y}$  in  $V_s$ ,

$$\partial_{y_j} \Sigma_{ji}^0 = 0, \quad (12a)$$

$$\Sigma_{ji}^0 = -p^0 \delta_{ji}, \quad (12b)$$

$$\partial_{y_i} U_i^0 = 0, \quad (12c)$$

for  $\mathbf{x}$  and  $\mathbf{y}$  in  $V_f$ , and on  $\partial V$ ,

$$U_i^0 = u_i^0, \quad (13a)$$

$$n_j \Sigma_{ji}^0 = n_j \sigma_{ji}^0. \quad (13b)$$

The next set of equations, of order  $\epsilon$ , is

$$\rho_s \partial_t^2 u_i^0 = \partial_{x_j} \sigma_{ji}^0 + \partial_{y_j} \sigma_{ji}^1, \quad (14a)$$

$$\sigma_{ji}^1 = \frac{\partial \sigma_{ji}(\partial_{x_m} u_n^0 + \partial_{y_m} u_n^1)}{\partial F_{pq}} (\partial_{x_p} u_q^1 + \partial_{y_p} u_q^2) \quad (14b)$$

in  $V_s$ ,

$$\rho_f \partial_t^2 U_i^0 = \partial_{x_j} \Sigma_{ji}^0 + \partial_{y_j} \Sigma_{ji}^1, \quad (15a)$$

$$\Sigma_{ji}^1 = -p^1 \delta_{ji} + 2 \eta \partial_t \{ \partial_{y_j} U_i^0 \}, \quad (15b)$$

$$\mathcal{Q}(p^0) = \partial_{x_i} U_i^0 + \partial_{y_i} U_i^1, \quad (15c)$$

in  $V_f$ , and

$$U_i^1 = u_i^1, \quad (16a)$$

$$n_j \Sigma_{ji}^1 = n_j \sigma_{ji}^1, \quad (16b)$$

on  $\partial V$ .

### C. Homogenization of the asymptotic equations

Introduce the relative fluid displacement vector  $w_i^0(\mathbf{x}, \mathbf{y}, t)$ , defined as

$$w_i^0(\mathbf{x}, \mathbf{y}, t) = \phi [U_i^0(\mathbf{x}, \mathbf{y}, t) - u_i^0(\mathbf{x}, t)], \quad (17)$$

where  $0 < \phi < 1$  is the porosity or volume fraction of the fluid,  $\phi = V_f/V$ . We can then rewrite Eq. (15c) as

$$-\mathcal{Q}(p^0) + \partial_{x_i} (u_i^0 + \phi^{-1} w_i^0) = -\partial_{y_i} U_i^1. \quad (18)$$

The dependence upon the fast scale may now be eliminated by averaging over  $\mathbf{y}$ . A formal definition of the averaging procedure can be given for nonperiodic, statistically defined media; see, for example, Burridge and Keller.<sup>7</sup> Here, for the sake of simplicity, we assume periodicity on the small scale, so that the average is trivial. Thus integrating Eq. (18) over  $V_f$ , using the displacement continuity conditions (16a) and the assumed periodicity in  $\mathbf{y}$ , gives

$$\begin{aligned} & \phi(-\mathcal{Q}(p^0) + \partial_{x_i} u_i^0) + V_f^{-1} \int_{V_f} \partial_{x_i} w_i^0 \, dV(\mathbf{y}) \\ &= (1 - \phi) V_s^{-1} \int_{V_s} \partial_{y_i} u_i^1 \, dV(\mathbf{y}), \quad \mathbf{x} \text{ in } V_f. \end{aligned} \quad (19)$$

Substitution of Eqs. (12b) and (15b) into Eq. (15a) yields

$$\begin{aligned} & \rho_f \partial_t^2 w_i^0 + \partial_{y_j} (p^1 \phi \delta_{ji} - 2 \eta \partial_t \{ \partial_{y_j} w_i^0 \}) \\ &= -\phi [\rho_f \partial_t^2 u_i^0 + \partial_{x_i} p^0], \quad \mathbf{x}, \mathbf{y} \text{ in } V_f, \end{aligned} \quad (20)$$

while Eqs. (12c) and (13a) become, respectively,

$$\partial_{y_i} w_i^0 = 0, \quad \mathbf{y} \text{ in } V_f, \quad (21a)$$

$$w_i^0 = 0, \quad \mathbf{y} \text{ on } \partial V. \quad (21b)$$

Next, integrating Eq. (14a) over  $V_s$  and Eq. (15a) over  $V_f$ , then adding the results and using the continuity conditions (16b) and periodicity, we find

$$\begin{aligned} & \rho \partial_t^2 u_i^0(\mathbf{x}, t) + \rho_f V_f^{-1} \int_{V_f} \partial_t^2 w_i^0 \, dV(\mathbf{y}) \\ &= -\phi \partial_{x_i} p^0 + (1 - \phi) V_s^{-1} \frac{\partial}{\partial_{x_j}} \int_{V_s} \sigma_{ji}^0 \, dV(\mathbf{y}), \end{aligned} \quad (22)$$

where

$$\rho = \phi \rho_f + (1 - \phi) \rho_s \quad (23)$$

is the average density. Finally, using Eq. (12b), we rewrite the boundary condition (13b) as

$$n_j \sigma_{ji}^0 = -p^0 n_i. \quad (24)$$

The system of equations (20) and (21) forms a closed boundary-value problem with respect to  $\partial_t w_i^0(\mathbf{x}, \mathbf{y}, t)$ , and in principle, one can express this function in terms of  $u_i^0$  and  $p^0$ . Similarly, Eqs. (11) and (13b) give a well defined boundary-value problem with respect to  $u_i^1$ , which enables us to present  $u_i^1$  as a function or functional of  $u_i^0$  and  $p^0$ . Both problems are solved explicitly for the case of a periodically layered medium in the next section. Inserting the results in Eqs. (19) and (22), we arrive at a closed master system of

equations with respect to  $u_i^0$ ,  $p^0$ , and  $w_i$ , with the averaged relative fluid displacement defined as

$$w_i(\mathbf{x}, t) \equiv V_f^{-1} \int_{V_f} w_i^0(\mathbf{x}, \mathbf{y}, t) dV(\mathbf{y}). \quad (25)$$

## II. CONSTITUTIVE THEORY FOR A LAYERED MEDIUM

We consider periodically alternating fluid and solid layers. The thickness of each fluid layer is  $l$  in terms of  $\mathbf{x}$  and  $L = l/\epsilon$  in terms of  $\mathbf{y}$ . Let  $y = \mathbf{y} \cdot \mathbf{n}$  be the coordinate in the direction of layering. The two microproblems defined at the end of the previous section then depend only upon  $y$ , and hence reduce to unidimensional problems. This is the great simplification that arises from the layering, and it is not present for any other configuration.

### A. The permeability operator

The permeability operator results from the solution of Eqs. (20) and (21). Let

$$v_i = \partial_t w_i^0, \quad f_i(\mathbf{x}, t) = -\rho_f \partial_t^2 u_i^0 - \partial_x p^0; \quad (26)$$

then Eqs. (20) and (21b) become

$$\partial_t v_i = -n_i \rho_f^{-1} \partial_y p^0 + \eta \rho_f^{-1} \partial_y^2 v_i + \phi f_i(\mathbf{x}, t), \quad (27a)$$

$$n_i \partial_y v_i = 0. \quad (27b)$$

We consider  $y$ -periodic solutions of this system. The boundary conditions are those for a rigid wall at  $y=0$  and  $y=L$ , which combined with the incompressibility condition for  $w_i^0$ , implies

$$n_i v_i = 0 \Leftrightarrow v_i = P_{ij} v_j, \quad (28)$$

where  $\mathbf{P}(\mathbf{n}) = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$  projects on to the horizontal plane. Contracting Eq. (27a) with  $\mathbf{n}$  and using Eq. (27b) allows us to eliminate the pressure gradient term. Bearing in mind Eq. (28) we deduce that  $v_i$  satisfies

$$\partial_t v_j - \nu \partial_y^2 v_j = \phi P_{ij} f_j(\mathbf{x}, t), \quad (29)$$

where  $\nu = \eta/\rho_f$  is the kinematic viscosity. This can be solved by standard means, i.e., using a Fourier series in  $y$ . A very similar type of problem is discussed by Sneddon.<sup>19</sup> Taking into account the initial data ( $v=0$  for  $t<0$ ), we find

$$v_i(\mathbf{x}, y, t) = \phi P_{ij} \sum_{n=1}^{\infty} \frac{2}{n} [1 - (-1)^n] \sin \frac{n\pi y}{L} \times \int_0^t d\xi e^{-\nu(n^2\pi^2/L^2)(t-\xi)} f_j(\mathbf{x}, \xi). \quad (30)$$

Averaging  $v_i(\mathbf{x}, y, t)$  over the layer yields  $\tilde{v}_i(\mathbf{x}, t)$ , which satisfies

$$\tilde{v}_i(\mathbf{x}, t) = K_{ij}(t) * f_j(\mathbf{x}, t), \quad K_{ij}(t) = P_{ij} K(t), \quad (31)$$

where  $*$  denotes convolution, and

$$K(t) = \phi \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \exp\left(-\pi^2(2n-1)^2 \frac{\nu t}{L^2}\right). \quad (32)$$

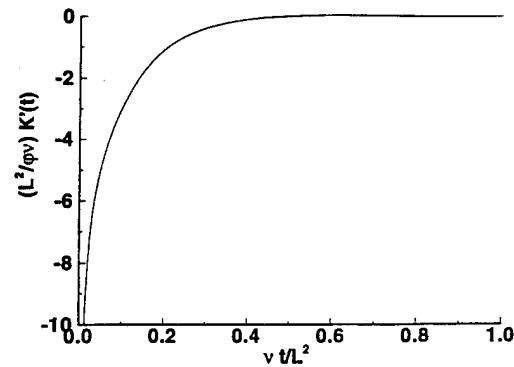


FIG. 1. The permeability function for the stratified medium in nondimensional units.

Hence the permeability of the medium is a projection operator onto the transverse plane. Also,  $K(t)$  is independent of  $\epsilon$ , because  $\nu/L^2 = \bar{\eta}/\rho_f l^2$ .

Note that Eq. (31) implies

$$\partial_t \tilde{v}_i(\mathbf{x}, t) = K_{ij}(0) f_j(\mathbf{x}, t) + K'_{ij}(t) * f_j(\mathbf{x}, t), \quad (33)$$

where  $K(0) = \phi$  for the layered medium, and  $K'(t)$  is plotted in Fig. 1. It follows from Eq. (32) as

$$K'(t) = -\phi \frac{8\nu}{L^2} \sum_{n=1}^{\infty} \exp\left(-\pi^2(2n-1)^2 \frac{\nu t}{L^2}\right). \quad (34)$$

This is a well behaved and convergent series, except for values of  $t$  approaching zero, where it is evidently singular. The behavior near zero can be seen by transforming the sum using the Poisson summation formula: for a given function  $g(x)$ ,

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) e^{-i2\pi mx} dx. \quad (35)$$

Applying this to the sum in Eq. (34) and performing the integration yields

$$K'(t) = -\phi \frac{4\nu}{L\sqrt{\pi\nu t}} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \exp\left(-\frac{n^2 L^2}{4\nu t}\right) \right\}. \quad (36)$$

The function  $K'(t)$  therefore has an integrable  $t^{-1/2}$  singularity at zero.

### B. The stress and strain in the solid layers

We now turn to the system defining the displacement  $u_i^1$  in terms of the macroscopic parameters  $u_i^0$  and  $p^0$ , Eqs. (11) and (24). Proceeding as we did for the previous microproblem, we look for the affine solution depending on the vertical coordinate  $y$  only. Hence

$$\partial_y u_i^1 = n_j G_i, \quad (37)$$

where the vector  $G_i$  does not depend upon  $y$ . Then  $u_i^1$  automatically satisfies the equilibrium equations within the solid region, Eq. (11). The traction boundary conditions of Eq. (24) become, using Eq. (11b),

$$n_j \sigma_{ji} (\partial_{x_m} u_n^0 + n_m G_n) = -p^0 n_i. \quad (38)$$

This is an algebraic system of three equations in three unknowns, with solution

$$G_n = G_n (\partial_{x_m} u_i^0, p^0). \quad (39)$$

Explicit versions of this will be discussed later, but for the moment the implicit solution of Eq. (39) is sufficient. The leading-order stress in the solid then follows from Eqs. (11b) and (37) as

$$\sigma_{ji}^0 = \sigma_{ji} (\partial_{x_m} u_n^0 + n_m G_n) \equiv T_{ji} (\partial_{x_m} u_n^0, p^0). \quad (40)$$

### III. MACROSCOPIC EQUATIONS

#### A. The general nonlinear equations

We are now in a position to state the leading-order macroscopic equations of motion. The fundamental field variables are the solid displacement  $u_i^0(\mathbf{x}, t)$  and the averaged fluid displacement  $w_i(\mathbf{x}, t)$ , and their governing dynamic equations follow from Eqs. (22), (26), and (31). In order to simplify the notation, we make the replacements  $u_i^0(\mathbf{x}, t) \rightarrow u_i(\mathbf{x}, t)$  and  $p^0(\mathbf{x}, t) \rightarrow p(\mathbf{x}, t)$ . The dynamic equations are

$$\rho \partial_t^2 u_i(\mathbf{x}, t) + \rho_f \partial_t^2 w_i(\mathbf{x}, t) = \tau_{ji,j}(\mathbf{x}, t), \quad (41a)$$

$$\partial_t w_i(\mathbf{x}, t) = -K_{ij}(t) * [\partial_t^2 u_j(\mathbf{x}, t) + \rho_f^{-1} p_{,j}(\mathbf{x}, t)], \quad (41b)$$

where  $K_{ij}(t)$  is given explicitly in Eq. (32). The stress and pressure are related to the displacements by the right-hand side of Eq. (22) and Eq. (40), and by Eqs. (19) and (37), respectively, or

$$\tau_{ji}(\mathbf{x}, t) = (1 - \phi) T_{ji}(\mathbf{F}, p) - \phi p(\mathbf{x}, t) \delta_{ij}, \quad (42a)$$

$$\zeta(\mathbf{x}, t) = \phi e(\mathbf{x}, t) - \phi \mathcal{Q}(p) - (1 - \phi) n_i G_i(\mathbf{F}, p), \quad (42b)$$

where  $F_{ij} = \partial_{x_j} u_i \equiv u_{i,j}$  is the macroscopic deformation gradient tensor,  $e \equiv u_{i,i}$ , and  $\zeta \equiv -w_{i,i}$  is the relative fluid dilatation.<sup>2</sup>

Equations (42) provide the bulk stress  $\tau_{ji}$  and the relative fluid dilatation  $\zeta$  in terms of the solid deformations  $F_{ij}$  and the fluid pressure  $p$ . When the pressure is zero, the bulk stress is

$$\tau_{ji} = (1 - \phi) T_{ji}(\mathbf{F}, 0), \quad p = 0. \quad (43)$$

Hence we can identify  $(1 - \phi) T_{ji}(\mathbf{F}, 0)$  as the stress function for the dry frame, or the "open" system.<sup>2</sup> If the pores are sealed, or "closed,"<sup>2</sup> then the associated stress function follows from Eqs. (42) with  $\zeta \equiv 0$ .

#### B. Energy potentials

The macroscopic elastic constitutive relation for the stress in Eq. (42a) can be related to the effective strain energy. Comparison of Eqs. (42) and (A13) implies that the former can be replaced by the simpler relations

$$\tau_{ji} = \frac{\partial W}{\partial F_{ji}}, \quad p = \frac{\partial W}{\partial \zeta}. \quad (44)$$

The effective strain energy follows from Eq. (A1), which becomes in the present notation

$$W(\mathbf{F}, \zeta) = (1 - \phi) E_s [F_{ji} + n_i G_j(\mathbf{F}, p)] + \phi E_f(\mathcal{Q}(p)). \quad (45)$$

Hence the effective medium is elastic in the sense that it possesses a stored energy potential  $W$ . Note that  $W$  is a function of the kinematic strains  $\mathbf{F}$  and  $\zeta$ , but its functional form is defined by  $\mathbf{F}$  and  $p$ . The remaining identity (42b) is an implicit relation for  $p$  in terms of  $\mathbf{F}$  and  $\zeta$ , i.e.,  $p = p(\mathbf{F}, \zeta)$ .

The related potential  $\Pi$  is defined in the Appendix. It follows from Eq. (A7) that the two constitutive relations of Eq. (42) are equivalent to

$$\tau_{ji} = \frac{\partial \Pi}{\partial F_{ji}}, \quad -\zeta = \frac{\partial \Pi}{\partial p}. \quad (46)$$

The functional definition of  $\Pi = \Pi(\mathbf{F}, p)$  is given in Eq. (A6), which becomes in the present notation

$$\begin{aligned} \Pi(\mathbf{F}, p) = & (1 - \phi) E_s (F_{ji} + n_i G_j(\mathbf{F}, p)) \\ & + (1 - \phi) \rho n_m G_m(\mathbf{F}, p) - \phi p e \\ & + \phi \int_{p_{\text{amb}}}^p \mathcal{Q}(\eta) d\eta. \end{aligned} \quad (47)$$

This is an explicit equation for the potential, apart from the fact that the functions  $G_i(\mathbf{F}, p)$  are given by the implicit relation (38). A simpler form for  $\Pi$  can be obtained by noting that it is a partial Legendre transform of the energy function  $W(\mathbf{F}, \zeta)$ . The differential relations (44) and (46) imply the connection

$$W = \Pi + \zeta p. \quad (48)$$

### IV. LINEAR THEORY FOR THE STRATIFIED MEDIUM

It is instructive to consider the limiting case for small amplitude waves, for which the linear constitutive equations in the solid and fluid are sufficient. The individual energy potentials are now

$$E_s(e_{ij}) = \frac{1}{2} C_{ijkl} e_{ij} e_{kl}, \quad E_f(e) = \frac{1}{2} \kappa_f e^2, \quad (49)$$

where  $\kappa_f$  is the fluid bulk modulus and the elastic moduli possess the symmetries  $C_{ijkl} = C_{klji}$  and  $C_{ijkl} = C_{jikl}$ . The linear stress in the solid is

$$\sigma_{ji}(u_{m,n}) = C_{jikl} u_{k,l}. \quad (50)$$

The deformation gradient tensor  $\mathbf{F}$  can be replaced in all expressions by the macroscopic linear strain tensor  $\mathbf{e} = (\mathbf{F} + \mathbf{F}^T)/2$ , or  $e_{ij} = (u_{i,j} + u_{j,i})/2$ . Equation (38) becomes a linear system for  $G_i$  which can be solved easily, yielding

$$G_i(\mathbf{e}, p) = -b_{ikl} e_{kl} - Q_{ik}^{-1} n_k p, \quad (51)$$

where

$$Q_{ik} = C_{ijkl} n_j n_l, \quad b_{ikl} = Q_{ip}^{-1} C_{pkl} n_j. \quad (52)$$

The functions  $T_{ji}$  of Eq. (40) then follow as

$$T_{ji}(\mathbf{e}, p) = (C_{ijkl} - b_{mij} b_{nkl} Q_{mn}) e_{kl} - b_{kij} n_k p. \quad (53)$$

Using the fact that  $\mathcal{Q}(p) = -p/\kappa_f$ , which follows from Eqs. (5) and (49), the constitutive relations of Eq. (42) become

$$\tau_{ji}(\mathbf{x}, t) = C_{ijkl}^0 e_{kl} - M^{-1} M_{ij} p, \quad (54a)$$

$$\zeta(\mathbf{x}, t) = M^{-1} M_{ij} e_{ij} + M^{-1} p, \quad (54b)$$

where

$$C_{ijkl}^0 = (1 - \phi)(C_{ijkl} - b_{mij}b_{nkl}Q_{mn}), \quad (55a)$$

$$M = [\phi/\kappa_f + (1 - \phi)n_i n_j Q_{ij}^{-1}]^{-1}, \quad (55b)$$

$$M_{ij} = [\phi \delta_{ij} + (1 - \phi)n_k b_{kij}]M. \quad (55c)$$

Note that  $C_{ijkl}^0$  are the dry frame or open pore moduli; see Eq. (43). Equations (54) are equivalent to

$$\tau_{ji} = C_{ijkl}^c e_{kl} - M_{ij} \zeta, \quad (56a)$$

$$p = M \zeta - M_{ij} e_{ij}, \quad (56b)$$

where

$$C_{ijkl}^c = C_{ijkl}^0 + M^{-1} M_{ij} M_{kl}, \quad (57)$$

are the “closed pore” moduli ( $\zeta \equiv 0$ ). Also,  $(1 - \phi)^{-1} C_{ijkl}^0$  can be identified as the moduli for a state of *plane stress* in the elastic solid. Thus Eq. (54a) indicates that macroscopic plane stress prevails when the pressure is zero, as expected.

The potential  $\Pi$  follows by integrating  $\tau_{ji}$  and  $\zeta$ , using Eqs. (46). The energy density then follows from Eq. (48). We find, after some simplification, that

$$W = \frac{1}{2} C_{ijkl}^0 e_{ij} e_{kl} + \frac{p^2}{2M}, \quad \Pi = \frac{1}{2} C_{ijkl}^c e_{ij} e_{kl} - \frac{M}{2} \zeta^2. \quad (58)$$

The former is a remarkably simple and physically appealing form. It clearly separates the strain energy into solid and fluid parts, where the latter depends upon the effective bulk modulus  $M$ .

## V. ISOTROPIC ELASTIC LAYERS AND WAVES

### A. The general equations

Let  $\lambda$  and  $\mu$  be the Lamé moduli of the solid; then Eqs. (52) become

$$\mathbf{Q}^{\pm 1} = (\lambda + 2\mu)^{\pm 1} \mathbf{n} \otimes \mathbf{n} + \mu^{\pm 1} \mathbf{P}(\mathbf{n}), \quad (59a)$$

$$b_{ikl} = \frac{\lambda}{\lambda + 2\mu} n_i P_{kl} + n_k P_{il} + n_l P_{ik} + n_i n_k n_l. \quad (59b)$$

It is a simple matter of algebra to show that the parameters in Eqs. (55) are now

$$C_{ijkl}^0 = (1 - \phi)(\lambda_0 P_{ij} P_{kl} + 2\mu \bar{I}_{ijkl}), \quad (60a)$$

$$M = \left( \frac{\phi}{\kappa_f} + \frac{1 - \phi}{\lambda + 2\mu} \right)^{-1}, \quad (60b)$$

$$M_{ij} = \left[ n_i n_j + \left( \frac{\lambda + 2\mu\phi}{\lambda + 2\mu} \right) P_{ij} \right] M, \quad (60c)$$

where  $\bar{I}_{ijkl} \equiv (P_{ik} P_{jl} + P_{il} P_{jk})/2$  is the “in-plane” fourth-order identity tensor and  $\lambda_0 = 2\mu\lambda/(\lambda + 2\mu)$  is the “plane stress” Lamé modulus. The two constitutive equations are (54a) and (54b), the former simplifying to

$$\tau_{ij} = (1 - \phi)(\lambda_0 \bar{e} P_{ij} + 2\mu \bar{e}_{ij}) - M^{-1} M_{ij} p, \quad (61)$$

where  $\bar{e}_{ij} = \bar{I}_{ijkl} e_{kl}$  is the in-plane strain, and  $\bar{e} = \bar{e}_{jj}$ . The constitutive relations (54b) and (61) are written with the solid strain  $e_{ij}$  and the fluid pressure  $p$  as the fundamental variables for the poroelastic medium. Alternatively, if we choose the strains  $e_{ij}$  and  $\zeta$  as the primitive variables, then the constitutive relations are Eqs. (56). The energy potential  $W$  of Eq. (58) reduces to

$$W = \frac{1}{2} (1 - \phi)(\lambda_0 \bar{e}^2 + 2\mu \bar{e}_{ij} \bar{e}_{ij}) + \frac{p^2}{2M}. \quad (62)$$

The Biot equations are those of a transversely isotropic poroelastic medium, as expected. The general Biot equations for a material with this symmetry<sup>20</sup> have eight independent moduli: five for the solid, one for the fluid, and two for the fluid/solid interaction. It is interesting that the present equations only have five independent moduli, two each for  $C_{ijkl}^0$  and  $M_{ij}$ , and one for  $M$ . Three of the eight possible constants are identically zero, which can be ascribed to the plane-stress configuration of the solid.

The equations of motion can be further simplified. Substituting  $\tau_{ij}$  from Eq. (61) into Eq. (41a) gives a decoupled system of equations. We now let  $\mathbf{n} = \mathbf{e}_3$ , where  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  form an orthonormal triad. Then, using Eq. (33), we find

$$\rho \partial_t^2 u_3 + p_{,3} = 0, \quad (63a)$$

$$\rho_s \partial_t^2 u_\alpha - \lambda_0 \bar{e}_{,\alpha} - 2\mu \bar{e}_{\alpha\beta,\beta} + (\lambda_0/2\mu) p_{,\alpha} = (1 - \phi)^{-1} K'(t) * (\rho_f \partial_t^2 u_\alpha + p_{,\alpha}), \quad (63b)$$

where the Greek suffices  $\alpha$  and  $\beta$  are restricted to 1 and 2, and the summation convention is implicit. Finally, eliminating  $\zeta$  between Eqs. (41b) and (56b) gives

$$\partial_t(p + M_{ij} e_{ij}) = M K(t) * (\partial_t^2 \bar{e} + \rho_f^{-1} p_{,\alpha\alpha}), \quad (64)$$

where  $K$  is given in Eq. (32). The four equations in Eqs. (63) and (64) form a closed set of equations of motion for  $\mathbf{u}(\mathbf{x}, t)$  and  $p(\mathbf{x}, t)$ . Next, we consider four examples of wave solutions in the layered medium.

### B. Examples of wave motion

#### 1. Propagation in the vertical direction

As a first example consider motion in the vertical direction only:  $\mathbf{u} = u(x_3, t) \mathbf{e}_3$  and  $\mathbf{w} = w(x_3, t) \mathbf{e}_3$ . Then Eq. (41b) implies  $w = 0$  while Eq. (42) gives  $\tau_{ji} = e_{33} M_{ji}$  and  $p = -e_{33} M$ . The equation of motion (63a) becomes simply

$$\rho \partial_t^2 u(x, t) - M \partial_x^2 u(x, t) = 0. \quad (65)$$

This is a scalar wave equation with wave speed  $\sqrt{M/\rho}$ .

#### 2. A damped shear wave

Consider shear motion in the horizontal plane:  $\mathbf{u} = u(x_1, t) \mathbf{e}_2$  and  $\mathbf{w} = w(x_1, t) \mathbf{e}_2$ . The dynamic equations again reduce to a single equation for  $u$ :

$$\rho_s \partial_t^2 u(x, t) - \mu \partial_x^2 u(x, t) = (1 - \phi)^{-1} K'(t) * \rho_f \partial_t^2 u(x, t). \quad (66)$$

The shear wave propagates with a time delayed memory function for the inertial term. The appearance of  $K'$  reflects

the influence of the fluid viscosity on the shear wave in the solid.

### 3. Wave motion when the fluid is inviscid: Schoenberg's problem

The limiting case of an inviscid fluid is relevant to experiments performed by Plona *et al.*<sup>3</sup> The fast and slow waves in the system are both nondispersive and nonattenuating in this limit. But the anisotropy of the configuration means that both wave types exhibit directional dependence, which can be best understood by considering the slowness surfaces. Schoenberg<sup>13,14</sup> derived the equations for the slowness surface from the dispersion relation for the long-wavelength modes of a layered system of fluid and isotropic solid. We will demonstrate that exactly the same slowness surface follows from the governing Biot equations outlined above.

The inviscid nature of the fluid means that the operator  $K$  is instantaneous, or  $K' \equiv 0$ . Differentiating Eq. (64) with respect to time then implies, using Eq. (33), that

$$\partial_t^2(p + M_{ij}e_{ij}) = MK(0)(\partial_t^2\bar{e} + \rho_f^{-1}p_{,\alpha\alpha}), \quad (67)$$

where  $K(0) = \phi$ . Consider motion in the  $x_1 - x_3$  plane with solutions of the form

$$\begin{aligned} u_1(t - s_1x_1 - s_3x_3), \quad u_2 = 0, \quad u_3(t - s_1x_1 - s_3x_3), \\ p = p(t - s_1x_1 - s_3x_3), \end{aligned} \quad (68)$$

where  $s_1$  and  $s_3$  are the horizontal and vertical components of the slowness. Let  $\mathbf{v} \equiv \partial_t \mathbf{u}$ ; then Eqs. (63) imply

$$\rho v_3 - s_3 p = 0, \quad (69a)$$

$$(1 - c_{pl}^2 s_1^2) v_1 - \sqrt{1 - c_{pl}^2 / c_p^2} s_1 p = 0, \quad (69b)$$

where  $c_p$  and  $c_{pl}$  are the wave speeds for longitudinal waves in the solid in bulk and as a thin plate, respectively. That is,

$$c_p^2 = (\lambda + 2\mu) / \rho_s, \quad c_{pl}^2 = (\lambda_0 + 2\mu) / \rho_s, \quad (70)$$

and therefore  $\lambda_0 / 2\mu = \sqrt{1 - c_{pl}^2 / c_p^2}$ . Eliminating  $v_3$  between Eqs. (67) and (69a) yields the third equation,

$$(1 - \phi) \sqrt{1 - c_{pl}^2 / c_p^2} s_1 v_1 - \left( \frac{1}{M} - \phi \frac{s_1^2}{\rho_f} - \frac{s_3^2}{\rho} \right) p = 0. \quad (71)$$

Equations (69b) and (71) are satisfied if  $s_1$  and  $s_3$  are related according to

$$\frac{s_3^2}{\rho_f} \left[ \frac{\phi}{\rho_f} (c_f^{-2} - s_1^2) + \frac{(1 - \phi)}{\rho_s (1 - c_{pl}^2 s_1^2)} (c_p^{-2} - s_1^2) \right] = 0, \quad (72)$$

where  $c_f = \sqrt{\kappa_f / \rho_f}$  is the speed of sound in the fluid. Equation (72) defines the slowness surface for the fast and slow waves, and it agrees with Schoenberg's<sup>13</sup> equation, derived by taking the low-frequency limit of the exact dispersion relation for a finely laminated medium of alternating solid and fluid layers (see also Refs. 3 and 14). We may rewrite Eq. (72) as

$$s_3^2 = \frac{\phi \rho}{\rho_f (c_{pl}^{-2} - s_1^2)} (s_{\text{fast}}^2 - s_1^2) (s_{\text{slow}}^2 - s_1^2), \quad (73)$$

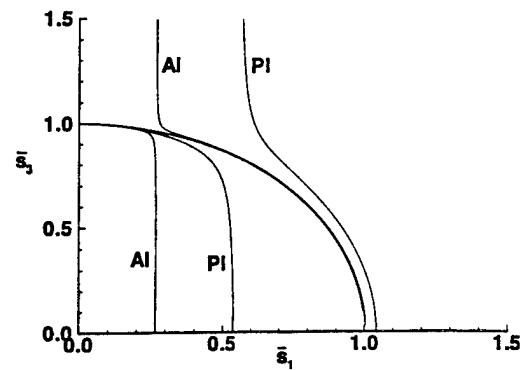


FIG. 2. The slowness surfaces for two different stratified media, consisting of water/aluminum (Al) and water/plexiglas (PI). The material parameters are in Table I and the porosity is  $\phi = 0.5$  in each case. The ordinate and abscissa are the dimensionless horizontal and vertical slownesses  $\bar{s}_1 = s_1 \sqrt{\phi M / \rho_f}$  and  $\bar{s}_3 = s_3 \sqrt{M / \rho}$ , respectively. The ellipse of Eq. (76), which becomes a circle in these units, is also shown. The plate slowness  $c_{pl}^{-1}$  corresponds to  $\bar{s}_1 = 0.268$  for aluminum and  $\bar{s}_1 = 0.561$  for plexiglas.

where  $s_{\text{fast}} < s_{\text{slow}}$  are the horizontal slowness for the fast and slow waves propagating horizontally. They satisfy

$$\frac{\phi}{\rho_f} (s_{\text{fast}}^2 + s_{\text{slow}}^2) = \frac{\phi}{\rho_f c_f^2} + \left( \frac{\phi}{\rho_f} + \frac{1 - \phi}{\rho_s} \right) \frac{1}{c_{pl}^2}, \quad (74a)$$

$$\frac{\phi}{\rho_f} s_{\text{fast}}^2 s_{\text{slow}}^2 = \left( \frac{\phi}{\rho_f c_f^2} + \frac{1 - \phi}{\rho_s c_p^2} \right) \frac{1}{c_{pl}^2}. \quad (74b)$$

These slowness values were also obtained by Rytov<sup>12</sup> who analyzed waves traveling horizontally in a fluid/solid layered medium.

Schoenberg<sup>13</sup> considered a Plexiglas/water system with  $\phi = 0.6$ , while Plona *et al.*<sup>3</sup> considered the same system with  $\phi = 0.5$ , and also aluminum/water with  $\phi = 0.5$ . In all cases it was found that  $s_{\text{fast}} < c_{pl}^{-1} < s_{\text{slow}}$ , and therefore the slowness surface of Eq. (73) comprises an inner closed sheet for  $0 \leq s_1 \leq s_{\text{fast}}$ , corresponding to the fast mode, and an outer sheet for  $c_{pl}^{-1} < s_1 \leq s_{\text{slow}}$  which gives the slow mode. A stop band exists for  $s_{\text{fast}} < s_1 < c_{pl}^{-1}$ . The value of  $s_{\text{fast}}$  is very close to  $c_{pl}$  in all these cases, and very accurate approximations can be obtained by iterating from this starting value; thus

$$s_{\text{slow}}^2 \approx \frac{1}{c_f^2} + \frac{(1 - \phi) \rho_f}{\phi \rho_s c_p^2} = \frac{\rho_f}{\phi M}, \quad (75a)$$

$$s_{\text{fast}}^2 \approx \frac{1}{c_{pl}^2} - \frac{(1 - \phi) \rho_f}{\phi \rho_s c_{pl}^2} \left( \frac{c_{pl}^{-2} - c_p^{-2}}{c_f^{-2} - c_p^{-2}} \right). \quad (75b)$$

Thus the slowness surface is essentially the ellipse

$$\frac{\phi s_1^2}{\rho_f} + \frac{s_3^2}{\rho} = \frac{1}{M}, \quad (76)$$

punctuated by a small stop band to the left of  $s_1 = c_{pl}^{-1}$ . The slowness surfaces for two material combinations are plotted in Fig. 2, which also shows the ellipse of Eq. (76).

The displacement polarization follows from Eqs. (69a). For a given value of  $s_1$ , we have

$$\frac{v_1}{p} = \frac{s_1}{\rho_s} \frac{\sqrt{1 - c_{pl}^2 / c_p^2}}{(1 - c_{pl}^2 s_1^2)}, \quad \frac{v_3}{p} = \frac{s_3}{\rho}, \quad (77)$$

where  $s_3$  is defined by Eq. (73). Assume that  $s_1 > 0$  and  $s_3 > 0$ , as in Fig. 2. The ratios  $v_1/p$  and  $v_3/p$  are both positive for the fast wave (the branch  $s_1 \leq s_{\text{fast}}$ ). The same ratios are negative and positive, respectively, for the slow wave. Thus the horizontal velocity is in phase with the acoustic fluid pressure for the fast wave, but out of phase for the slow wave.

#### 4. Dynamic compatibility

The phenomenon of dynamic compatibility was noted by Biot in his first paper on the dynamics of isotropic porous media.<sup>1</sup> If the material parameters satisfy a certain constraint, then a wave solution exists which has no viscous attenuation and its relative fluid motion is zero, or  $w=0$ . Analogous compatibility conditions exist for the layered medium, as we now demonstrate for horizontal wave motion. Consider longitudinal motion in the horizontal plane,  $\mathbf{u}=u(x_1, t)\mathbf{e}_1$  and  $\mathbf{w}=w(x_1, t)\mathbf{e}_1$ . This leads to a pair of coupled equations for  $u(x, t)$  and  $w(x, t)$ ,

$$\begin{aligned} \rho \partial_t^2 u - [(1-\phi)(\lambda_0 + 2\mu) + Mq^2] \partial_x^2 u + \rho_f \partial_t^2 w \\ - Mq \partial_x^2 w = 0, \end{aligned} \quad (78a)$$

$$\partial_t w + K(t) * \left[ \partial_t^2 u - \frac{M}{\rho_f} \partial_x^2 (qu + w) \right] = 0, \quad (78b)$$

where  $q = (\lambda + 2\mu\phi)/(\lambda + 2\mu)$ . The system reduces to

$$\rho_f \partial_t^2 u - Mq \partial_x^2 u = 0, \quad w = 0, \quad (79)$$

if the physical parameters are related by

$$Mq/\rho_f = [(1-\phi)(\lambda_0 + 2\mu) + Mq^2]/\rho, \quad (80)$$

or equivalently, if the densities are in the ratio

$$\frac{\rho_s}{\rho_f} = \frac{\lambda_0}{2\mu} + \left[ \frac{\lambda_0 + 2\mu}{\lambda_0(1-\phi) + 2\mu\phi} \right] \frac{2\mu}{M}. \quad (81)$$

This is the condition of *dynamic compatibility*, and when it is satisfied waves can propagate unattenuated by viscous drag with wave speed  $\sqrt{Mq/\rho_f}$ . Alternatively, the right-hand side of Eq. (81) may be written in terms of the bulk compressional sound speeds and Poisson's ratio of the solid,  $\nu_s = \lambda/2(\lambda + \mu)$ , yielding

$$\frac{\rho_s}{\rho_f} = \frac{1 - \nu_s - \phi(1 - 2\nu_s)}{\nu_s - \phi(1 - 2\nu_s)[(1 - \nu_s)^{-1}(c_p^2/c_f^2) - 1]}. \quad (82)$$

The right-hand member is an increasing function of  $\phi$  for most material combinations, i.e.,  $c_p^2/c_f^2(1 - \nu_s) - 1 > 1$  normally. Therefore, for a given pair of materials, there is a unique value of the porosity at which compatibility is attained as long as  $\rho_s/\rho_f > (1 - \nu_s)/\nu_s$ .

#### VI. CONCLUSION

The theory of homogenization has been used to derive the nonlinear equations for a fluid infiltrated solid skeleton starting from the fundamental equations governing the motion on the microscale of the pores and grains. The canonical microgeometry of a layered system of alternating solid and fluid constituents has been analyzed in detail. The main re-

sults are summarized in the "averaged" or macroscopic equations of motion (41), and the nonlinear constitutive relations of Eqs. (42). The viscous effects of the pore fluid are contained in the permeability convolution operator defined by  $K_{ij}(t)$  of Eq. (32). The stress-strain relations in Eqs. (42) allow for arbitrary nonlinear behavior in the elastic solid and in the fluid. The key quantity is the vector function  $G_i$  defined by Eqs. (38) and (39). We have also shown how the stress and strain are related to a macroscopic potential energy, in the spirit of Biot's later work on nonlinear mechanics of porous media.<sup>21</sup>

The linear limit of these equations is of interest, and we have derived for the first time in Secs. IV and V the full set of linear Biot equations for a layered medium. The predicted form of the slowness surface for isotropic elastic layers agrees with that obtained by Schoenberg<sup>13,14</sup> from the exact dispersion relation for the system. This gives us confidence in the general validity of the nonlinear Biot theory as derived here.

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#### APPENDIX: ENERGY CONSIDERATIONS

The leading-order term for the macroscopic energy density,  $W$ , in the porous medium is a linear combination of the energy in the solid and the energy in the fluid, each averaged over its domain. The fluid energy follows from Eq. (15c) as  $E_f[\mathcal{Q}(p^0)]$ , which is independent of  $\mathbf{y}$ , so that the sum of the average partial energy is

$$W = V^{-1} \int_{V_s} E_s(\partial_{x_m} u_n^0 + \partial_{y_m} u_n^1) dV(\mathbf{y}) + \phi E_f[\mathcal{Q}(p^0)]. \quad (A1)$$

The purpose of this appendix is to convince the reader that  $W$  of Eq. (A1), considered as a function of the independent macroscopic strain variables, does indeed yield the stress and pressure as partial derivatives according to Eq. (44). We follow Sanchez-Palencia<sup>9</sup> and first consider the function

$$\begin{aligned} \Lambda(\partial_{x_m} u_n^0, p^0) = V^{-1} \int_{V_s} E_s(\partial_{x_m} u_n^0 + \partial_{y_m} u_n^1) dV(\mathbf{y}) \\ + p^0 V^{-1} \int_{\partial V_s} u_i^1 n_i dS(\mathbf{y}). \end{aligned} \quad (A2)$$

The point is that  $\Lambda$  is clearly a function of the arguments shown, because the field variables  $u_i^1$  are implicit functions of both  $\partial_{x_m} u_n^0$  and  $p^0$ , as discussed above [see Eqs. (11) and (13b)]. The first variation is

$$\begin{aligned} \delta\Lambda = V^{-1} \int_{V_s} \frac{\partial E_s(\partial_{x_m} u_n^0 + \partial_{y_m} u_n^1)}{\partial \Gamma_{ij}} \\ \times (\partial_{x_i} \delta u_j^0 + \partial_{y_i} \delta u_j^1) dV(\mathbf{y}) \end{aligned}$$

$$+ \delta p^0 V^{-1} \int_{\partial V_s} u_i^1 n_i dS(\mathbf{y}) + p^0 V^{-1} \int_{\partial V_s} \delta u_i^1 n_i dS(\mathbf{y}). \quad (\text{A3})$$

The terms involving  $\delta u_i^1$  annihilate one another on account of Eqs. (12b), (13b), and the first of (5). We are then left with

$$\begin{aligned} \delta\Lambda &= \delta(\partial_{x_m} u_n^0) V^{-1} \int_{V_s} \sigma_{mn}^0 dV(\mathbf{y}) \\ &+ \delta p^0 V^{-1} \int_{V_s} \partial_{y_i} u_i^1 dV(\mathbf{y}). \end{aligned} \quad (\text{A4})$$

Thus, referring to Eqs. (37) and (40), we have

$$\begin{aligned} (1 - \phi) n_i G_i(\partial_{x_m} u_n^0, p^0) &= \frac{\partial \Lambda}{\partial p^0}, \\ (1 - \phi) T_{ji}(\partial_{x_m} u_n^0, p^0) &= \frac{\partial \Lambda}{\partial (\partial_{x_j} u_i^0)}, \end{aligned} \quad (\text{A5})$$

respectively.

Consider the related function

$$\begin{aligned} \Pi(\partial_{x_m} u_n^0, p^0) &= \Lambda(\partial_{x_m} u_n^0, p^0) - \phi p^0 \partial_{x_i} u_i^0 \\ &+ \phi \int_{p_{\text{amb}}}^{p^0} \mathcal{Q}(\eta) d\eta, \end{aligned} \quad (\text{A6})$$

where  $p_{\text{amb}}$  is the ambient pressure.  $\Pi$ , like  $\Lambda$ , is a function of the independent variables  $\partial_{x_m} u_n^0$  and  $p^0$ . Taking its partial derivatives with respect to these, and using the identities of Eqs. (A5) and (42), we obtain the identities of Eq. (46). Thus

$$\delta\Pi(\mathbf{F}, p) = \tau_{ji} \delta F_{ji} - \zeta \delta p \quad (\text{A7})$$

or

$$\delta\tilde{\Pi}(\mathbf{F}, \zeta) = \tau_{ji} \delta F_{ji} + p \delta \zeta, \quad (\text{A8})$$

where

$$\tilde{\Pi}(\mathbf{F}, \zeta) = \Pi(\mathbf{F}, p) + \zeta p. \quad (\text{A9})$$

We will now demonstrate that  $\tilde{\Pi}$  and  $W$  are one and the same. The latter can be rewritten using Eq. (15c) as

$$\begin{aligned} W &= V^{-1} \int_{V_s} E_s(\partial_{x_m} u_n^0 + \partial_{y_m} u_n^1) dV(\mathbf{y}) \\ &+ V^{-1} \int_{V_f} E_f(\partial_{x_m} U_m^0 + \partial_{y_m} U_m^1) dV(\mathbf{y}). \end{aligned} \quad (\text{A10})$$

The first variation is

$$\begin{aligned} \delta W &= V^{-1} \int_{V_s} (\partial_{x_i} \delta u_j^0 + \partial_{y_i} \delta u_j^1) \sigma_{ij}^0 dV(\mathbf{y}) \\ &- V^{-1} \int_{V_f} p^0 \delta \mathcal{Q}(p^0) dV(\mathbf{y}), \end{aligned} \quad (\text{A11})$$

where Eqs. (15c) and the second of (5) have been used. The integral over the fluid region can be simplified further using Eqs. (16a), (18), and integration by parts, giving

$$\begin{aligned} \delta W &= V^{-1} \int_{V_s} (\partial_{x_i} \delta u_j^0 + \partial_{y_i} \delta u_j^1) \sigma_{ij}^0 dV(\mathbf{y}) \\ &- \phi p^0 \delta(\partial_{x_i} u_i^0) - p^0 \delta(\partial_{x_i} w_i) \\ &+ p^0 V^{-1} \int_{\partial V_s} \delta u_i^1 n_i dS(\mathbf{y}). \end{aligned} \quad (\text{A12})$$

The terms with  $\delta u_i^1$  again disappear because of Eqs. (12b), (13b), and the first of (5), leaving

$$\begin{aligned} \delta W &= \delta(\partial_{x_m} u_n^0) \left( V^{-1} \int_{V_s} \sigma_{mn}^0 dV(\mathbf{y}) - \phi p^0 \delta_{mn} \right) \\ &- p^0 \delta(\partial_{x_i} w_i). \end{aligned} \quad (\text{A13})$$

This implies the identities of Eq. (44), and the equivalence

$$\tilde{\Pi} = W. \quad (\text{A14})$$

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